GRADUATE STUDIES IN MATHEMATICS 182

Applied Mathematics

Nonlinear PDEs

A Dynamical Systems Approach

Guido Schneider
Hannes Uecker

American Mathematical Society

Nonlinear PDEs

A Dynamical Systems Approach

Guido Schneider
Hannes Uecker

GRADUATE STUDIES IN MATHEMATICS 182

Nonlinear PDEs

A Dynamical Systems Approach

Guido Schneider
Hannes Uecker

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification*. Primary 35-01, 35Bxx, 35Qxx, 37Kxx, 37Lxx.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-182

Library of Congress Cataloging-in-Publication Data

Names: Schneider, Guido, 1964– | Uecker, Hannes, 1970–

Title: Nonlinear PDEs : a dynamical systems approach / Guido Schneider, Hannes Uecker.

Other titles: Nonlinear partial differential equations

Description: Providence, Rhode Island : American Mathematical Society, [2017] | Series: Graduate studies in mathematics ; volume 182 | Includes bibliographical references and index.

Identifiers: LCCN 2017012328 | ISBN 9781470436131 (alk. paper)

Subjects: LCSH: Differential equations, Partial. | Differential equations, Nonlinear. | AMS: Partial differential equations – Instructional exposition (textbooks, tutorial papers, etc.). msc | Partial differential equations – Qualitative properties of solutions – Qualitative properties of solutions. msc | Partial differential equations – Equations of mathematical physics and other areas of application – Equations of mathematical physics and other areas of application. msc | Dynamical systems and ergodic theory – Infinite-dimensional Hamiltonian systems – Infinite-dimensional Hamiltonian systems. msc | Dynamical systems and ergodic theory – Infinite-dimensional dissipative dynamical systems – Infinite-dimensional dissipative dynamical systems. msc

Classification: LCC QA377 .S35584 2017 | DDC 515/.353–dc23 LC record available at `https://lccn.loc.gov/2017012328`

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1   22 21 20 19 18 17

To Daniela, Max and Jonas
and Anja, Franka and Henrike

Contents

Preface

"If you want to build a ship, don't herd people together to collect wood and don't assign them tasks and work, but rather teach them to long for the endless immensity of the sea"

ANTOINE DE SAINT-EXUPERY

This is an introductory textbook about nonlinear dynamics of partial differential equations (PDEs), with a focus on problems over unbounded domains and modulation equations. We explain how dynamical systems methods can be used to analyze PDEs in order to get more insight into the real world phenomena behind the equations. Our presentation is example-oriented and the starting point is very often a real world problem. This means that new mathematical tools are developed step by step in order to analyze the equations. They are re-applied and improved in subsequent sections to handle more and more complicated systems. In the end the reader should have learned mathematical tools for the analysis of some important classes of nonlinear PDEs and gained insight into nonlinear dynamics phenomena which may occur in PDEs.

The book is divided into four parts. In order to keep the book as an introductory text and as self-contained as possible, Part I is an introduction into finite-dimensional dynamics, defined by ordinary differential equations (ODEs), including bifurcation theory, attractors, and the basics of Hamiltonian dynamics. In Part II we explain the major differences between finitely and infinitely many dimensions and that in principle a PDE on a bounded domain is isomorphic to a system of countably many ODEs. We give two main applications of this point of view. The first one is the characterization of the attractor for the Allen-Cahn equation on an interval, which is also known as the Chafee-Infante problem. The second one is a very basic introduction to the Navier-Stokes equations, with a focus on periodic boundary conditions.

Genuine PDE phenomena such as transport, diffusion, and dispersion can hardly be understood by the interpretation of PDEs as systems of infinitely many ODEs. In Part III we consider PDEs which are posed on the real line. We start with the linear heat equation, and then turn to nonlinear problems. For famous model equations such as the Kolmogorov-Petrovsky-Piskounov or Fisher equation, the Korteweg-de Vries (KdV) equation, the Nonlinear Schrödinger (NLS) equation, and the Ginzburg-Landau (GL) equation, we discuss the local existence and uniqueness of solutions, special solutions as fronts and pulses, their stability and instability, soliton dynamics, the construction of attractors, and some related results.

The equations from Part III all play an important role in mathematics and have entire monographs devoted to each. Moreover, they have many connections to physics and other fields of applications, where they are often used as simplest possible models for the description of some real world phenomena. In Part IV we explore these connections from a mathematical perspective. The scalar equations from Part III occur as asymptotic effective models, or more specifically as modulation equations, for the more complicated systems from physics considered in Part IV. Examples are pattern forming systems which can be described by the GL equation, light pulses in nonlinear optics which can be described by the NLS equation, or long waves in dispersive systems which can be described by the KdV equation. We discuss how the dynamics of the reduced model equations transfer to the more complicated systems. Thus, in Part IV we give a mathematically rigorous presentation of the formalism of modulation equations in the context of real world applications. While this last part is close to recent research, it is still in textbook style, and often we do not prove the sharpest or most general result possible, but instead refer to the literature for extensions.

All chapters are kept as self-contained as possible, such that the reader can start to read directly about his or her favorite equation. Having a good background in linear ODEs, cf. §2.1, a starting point for our goals and objectives are §2.2-§2.3 about basic nonlinear ODE dynamics combined with Part III. There are other possible combinations, for instance the sections about dissipative dynamics or the sections about conservative dynamics. Nevertheless the reader can also read the book from the beginning to the end. See the Grasshopper's Guide on page 12 for detailed proposals. All chapters contain exercises which we strongly recommend not to skip.

This book grew out of our manuscripts for the lectures and seminars we gave about ODEs and PDEs at the universities of Bayreuth, Karlsruhe, Oldenburg, and Stuttgart. We thank the students who attended our lectures and seminars and urged us to keep the presentation simple and accessible. Moreover, we thank all friends and colleagues with whom we cooperated over the years, mainly on topics from Part IV, in particular, Dirk Blömker, Tom Bridges, Kurt Busch, Martina Chirilus-Bruckner, Christopher Chong, Walter Craig, Markus Daub, Hannes de Witt, Arjen Doelman, Tomas Dohnal, Wolf-Patrick Düll, Wiktor Eckhaus, Jean-Pierre Eckmann, Bernold Fiedler, Thierry Gallay, Dieter Grass, Daniel Grieser, Mark Groves, Tobias Häcker, Mariana Haragus, Ronald Imbihl, Ralf Kaiser, Tasso Kaper, Klaus Kirchgässner, Markus Kunze, David Lannes, Vincent Lescarret, Karsten Matthies, Ian Melbourne, Andreas Melcher, Johannes Müller, Robert Pego, Dmitry Pelinovsky, Jens Rademacher, Björn Sandstede, Arnd Scheel, Zarif Sobirov, Aart van Harten, C. Eugene Wayne, Daniel Wetzel, Peter Wittwer, and Dominik Zimmermann. We thank Stefanie Siegert and the unknown referees for a number of additional proposals to improve the presentation. Especially we thank Alexander Mielke from whom we learned about nonlinear dynamics and PDEs.

Finally we would like to thank Ina Mette from the AMS for her never ending motivation to go on with this book project and many helpful comments to transform our lecture notes into a book.

Guido Schneider and Hannes Uecker
Stuttgart and Oldenburg, February 2017

Introduction

Mathematicians want to classify things. However, with partial differential equations (PDEs) they had to stop on a rather unsatisfactory level. The reason for this is that almost all rules of theoretical physics and engineering, and many rules in life sciences and economics, are formulated as ordinary or partial differential equations (ODEs or PDEs). As different as the applications of differential equations are, as different is the behavior of their solutions. Therefore, a mathematical theory which wants to cover all differential equations can only cover the absolute basics. Hence, the books about PDEs necessarily differ strongly by the choice of examples and by the choice of mathematical theory which will be applied to the examples. There are entire books only covering one special important equation. Before we give the goals and objectives of this book we start with a short review of three important examples.

1.1. The three classical linear PDEs

In many courses about PDEs the following three examples, namely the Laplace equation, the heat equation, and the wave equation, play a major, sometimes exclusive, role.

Example 1.1.1. The Laplace equation is an equation for an unknown function $u : \Omega \to \mathbb{R}$ of two or more variables $x = (x_1, \ldots, x_d) \in \Omega \subset \mathbb{R}^d$ in terms of certain of its partial derivatives, namely

$$\Delta u = 0, \tag{1.1}$$

where $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_d}^2$. This PDE plays an important role in mathematics since the real and imaginary part of an analytic function in the complex plane satisfy the Laplace equation. It also plays a major role in applications. For instance the potential of an irrotational flow of an incompressible fluid such as water, or a stationary temperature field, or the potential of a stationary electric field in the absence of charges in Ω, satisfy this equation.

In order to solve this equation uniquely in a domain Ω, additional conditions are needed. To gain an intuition for the required boundary conditions we consider the factors which should determine a stationary temperature field u in a room $\Omega \subset \mathbb{R}^3$ as sketched from the side in Figure 1.1. The temperature will be determined by the temperature at the walls, the windows, the doors and the heating of the room, mathematically speaking by the conditions at the boundary $\partial\Omega$ of Ω.

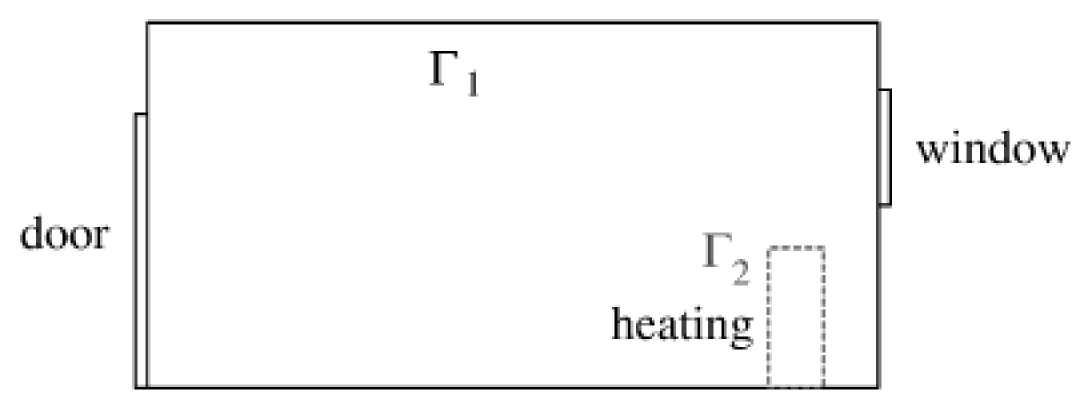


Figure 1.1. Different boundary conditions for the temperature field.

There are mainly two different kinds of boundary conditions. At the heating unit the temperature has a fixed value, while at a window or wall heat will go through the window or wall. Mathematically speaking the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is split into two parts where in the first part we have so called Dirichlet conditions

$$u|_{\Gamma_1} = g_1,$$

and in the second part we have so called Neumann conditions

$$\partial_n u|_{\Gamma_2} = g_2,$$

with given functions $g_1 : \Gamma_1 \to \mathbb{R}$ and $g_2 : \Gamma_2 \to \mathbb{R}$ and $n : \partial\Omega \to \mathbb{R}^3$ the outer normals.

The Laplace equation is the paradigm of an **elliptic** PDE. It is of second order, i.e., the highest derivative is of order two. There is an extensive theory for elliptic systems, especially for second order elliptic systems. The equilibrium equation of linear elasticity

$$Lu := \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u) \stackrel{!}{=} 0,$$

for the displacement vector $u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3$, with constants $\lambda, \mu \in \mathbb{R}$ depending on the material, is also a second order elliptic system. Like the negative Laplace operator $-\Delta$, the linear operator $-L$ defined in this equation is an example of a so called elliptic operator. Due to the important role of elasticity in the construction of cars, bridges, planes, etc., there are well developed numerical schemes such as the finite element method **(FEM)** or the boundary element method **(BEM)**, which are available for computing approximate solutions of such systems. Only in very special cases solutions can be found analytically. ⌋

Example 1.1.2. The heat equation or diffusion equation

$$\partial_t u = \Delta u, \tag{1.2}$$

with $u = u(x,t)$ and $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, where t denotes time and x denotes space, describes the evolution of quantities such as heat, chemical concentrations, or the probability distribution of a particle obeying Brownian motion.

It can be derived as follows. Let $V \subset \Omega$ be an arbitrary subset with smooth boundary. The change of the total quantity within V equals the flux through ∂V, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u \,\mathrm{d}x = -\int_{\partial V} \langle F, n\rangle \,\mathrm{d}S = -\int_V \nabla \cdot F \,\mathrm{d}x$$

with F the flux density, $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^d$, $n : \partial\Omega \to \mathbb{R}^d$ again the outer normal, and where we used the Gauss' integral theorem. Since this relation is true for all sets V, we find

$$\partial_t u = -\nabla \cdot F.$$

Very often the flux density F is proportional to the gradient ∇u of the concentration u, i.e., $F = -a\nabla u$ with a constant $a > 0$. By rescaling time we finally come to the diffusion equation (1.2).

In order to solve this equation uniquely in a domain $\Omega \times \mathbb{R}^+$ additional conditions are needed. As in Example 1.1.1 we need boundary conditions, but also the temperature field at time $t = 0$ has to be known, i.e., we need the initial condition $u|_{t=0} = u_0$ with $u_0 : \Omega \to \mathbb{R}$. Stationary solutions, i.e., time-independent solutions, satisfy the Laplace equation (1.1). The heat equation is the prototype **parabolic** PDE. There is an extensive theory for equations of the form $\partial_t u = Lu$ with an elliptic operator $-L$. ⌋

Example 1.1.3. The linear wave equation

$$\partial_t^2 u = \Delta u, \tag{1.3}$$

$u = u(x,t)$ with $u : \Omega \times \mathbb{R} \to \mathbb{R}$, is a simple model for, e.g., oscillations of a string ($\Omega \subset \mathbb{R}$) or of a membrane ($\Omega \subset \mathbb{R}^2$), or the propagation of light in vacuum. In order to solve this equation uniquely in a domain $\Omega \times (t_0, t_1)$, $t_0 < 0 < t_1$ we again need boundary and initial conditions. Like for scalar second order ODEs we need two initial conditions, namely $u|_{t=0} = u_0$ and $\partial_t u|_{t=0} = u_1$ with $u_0 : \Omega \to \mathbb{R}$ and $u_1 : \Omega \to \mathbb{R}$. The Dirichlet boundary condition $u|_{\partial\Omega}$ corresponds to a membrane which is fixed at the boundary. In this case, the boundary will reflect the waves.

For the wave equation the eigenmodes play a crucial role. An eigenmode is a solution $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t} v(x)$. This yields the eigenvalue problem

$$-\Delta v = \omega^2 v.$$

Such problems play an important role in applications, especially in elasticity theory, where the evolution equations of linear elasticity

$$\partial_t^2 u = \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u)$$

yield to the eigenvalue problem

$$-\mu \Delta v + (\lambda + \mu) \nabla (\nabla \cdot v) = \omega^2 v.$$

If Ω is a bounded set then under suitable boundary conditions there are countably many real eigenvalues $\lambda_n = \omega_n^2$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. In the construction of cars, bridges, planes, etc., one has to take care that these so called resonant modes are not periodically excited. Hence, there is a big industry using FEM and BEM in order to solve these elliptic eigenvalue problems. The wave equation is the prototype **hyperbolic** PDE. There is an extensive theory for equations of the form $\partial_t^2 u = Lu$ with an elliptic operator $-L$. ⌋

For reasons explained below we will focus on other examples than the three classical ones. The fundamental Examples 1.1.1-1.1.3 cannot be and will not be avoided. However, they will only occur as subproblems which will help to understand the nonlinear problems under consideration.

1.2. Nonlinear PDEs

We now start discussing our main objectives for this book, namely an introduction to nonlinear PDEs from a dynamical systems point of view, with a focus on reduction methods, in particular, the use of amplitude and modulation equations.

Many complications with ODEs or PDEs are due to the fact that **the world is nonlinear.** Ultimately, to solve a PDE means to look for solutions u of an abstract equation $F(u) = 0$. The problem is called linear if for all $\alpha, \beta \in \mathbb{R}$ we have

$$F(\alpha u + \beta v) = \alpha F(u) + \beta F(v).$$

As a consequence, for linear problems we have the superposition of solutions. With u, v being solutions, i.e., $F(u) = 0$ and $F(v) = 0$, also $\alpha u + \beta v$ is a solution, i.e., $F(\alpha u + \beta v) = 0$. Most “real life” problems are nonlinear, i.e., in general

$$F(\alpha u + \beta v) \neq \alpha F(u) + \beta F(v),$$

and therefore a sum of two solutions is no longer a solution of the ODE or PDE. A simple example of a nonlinear function is $F(u) = u^2$. As a consequence, the theory of linear algebra is not available, and the set of solutions in general is more complicated than that for linear problems. In science, for many decades linear problems played a dominating role. Examples 1.1.1 to 1.1.3 are linear. Next we present two famous examples of nonlinear PDEs.

Example 1.2.1. The Navier-Stokes equations

$$\begin{aligned} \partial_t u =& \frac{1}{R}\Delta u - \nabla p - (u \cdot \nabla)u, \\ 0 =& \nabla \cdot u, \end{aligned}$$

describe the evolution of the velocity field $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ and the pressure field $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ of an incompressible fluid, such as water, in a domain $\Omega \subset \mathbb{R}^3$. The Reynolds number R measures the ratio between inertial and viscous forces, and is in some sense proportional to the complexity of the flow. The global existence and uniqueness of smooth solutions of the three-dimensional (3D) Navier-Stokes equations is one of the seven one million dollar or Millennium problems in mathematics presented by the Clay-Foundation in the year 2000. There are a number of reasons for this choice. On the one hand, the Navier-Stokes equations describe the motion of fluids, and the answer to this question would allow us to understand fluids in a much better way. On the other hand, in mathematics the 3D Navier-Stokes equations are interesting PDEs, which so far have resisted all attempts to prove the global existence and uniqueness of solutions. This will be explained in more detail in Chapter 6. ⌋

Example 1.2.2. Maxwell's equations in a medium, for instance a glass fiber, are given by

$$\begin{aligned} \nabla \cdot B =& 0, \\ \nabla \times E + \partial_t B =& 0, \\ \nabla \cdot D =& \rho, \\ \nabla \times H - \partial_t D =& J. \end{aligned}$$

Here $E : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is the electric field, $D = \varepsilon_0 E + P$ is the displacement field, with ε_0 the electric permeability of vacuum, $P : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is the electric polarization of the material, $B : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is the magnetic field, $H = B/\mu_0 - M$ is the magnetizing field, with μ_0 the magnetic permeability of vacuum and $M : \Omega \times \mathbb{R} \to \mathbb{R}^3$ the magnetic polarization of the material, ρ is the charge density, and $J : \Omega \to \mathbb{R}^3$ the charge flow density. Since the first and the third equation above are scalar, while the second and fourth equation are vector valued, so far we have eight equations for the twelve unknowns B_j, E_j, M_j and P_j for $j = 1, 2, 3$. Therefore, these equations have to be closed with constitutive laws $P = P(E, H)$ and $M = M(E, H)$ describing the reaction of the material to the electric and magnetic field. In general, these laws are nonlinear. Moreover, as an additional complication M and P may depend on the past, cf. §11.7. $\rfloor$

The world is instationary, i.e., almost all systems evolve with time. Typical examples are a vibrating beam, the daily change of the weather, or the motion of the planets in the solar system. Hence, from the beginning we will consider **nonlinear time-dependent systems**.

A mathematical concept which is basic to the analytical understanding of all ODEs and PDEs is the concept of **Dynamical Systems**. Until the beginning of the 1960s, Laplace's principle that with the knowledge of all physical rules and the present state of the world, the past and future behavior of the world for all times can be computed, was widely accepted as a relevant philosophical foundation of science. Starting already with the work of H. Poincaré in the 1890s, cf. [**Poi57**], this principle was finally observed to be practically useless at the beginning of the 1960s, for instance by the work of the meteorologist E. Lorenz in 1963 [**Lor63**]. He observed with an analog computer for a three-dimensional model for the weather that the possible time for predictions goes logarithmically with the precision of the initial conditions, i.e., that long-time weather-forecasts are practically impossible. See Figure 1.2 for an illustration of the so called Lorenz attractor and of the sensitivity of solutions w.r.t. the initial conditions.

Certain ODEs and PDEs, or, more general, dynamical systems, can be classified as **chaotic**. The visualization of chaotic dynamical systems was in fashion in the 1980s. Famous examples are the Mandelbrot and the Julia sets. In this book, chaos will not play a central role, but one should keep in mind its existence already in low-dimensional dynamical systems.

1.3. Our choice of equations and the idea of modulation equations

PDEs play an important role in modern engineering. With the help of computer simulations, money can be saved, experiments can be replaced, and data can be gathered which are not available by classical experiments. However, **a numerical simulation of a PDE requires an analytic understanding of the PDE.** The reason for this is again the wide variety of different types of PDEs. Therefore, very often the numerical simulation of a PDE needs an adapted numerical scheme based on an analysis of the PDE. As the example of the crash of the Sleipner oil platform in 1991 shows, a misuse of numerical schemes can cost a lot of money. In the concrete example 700 million dollars [**JR94**].

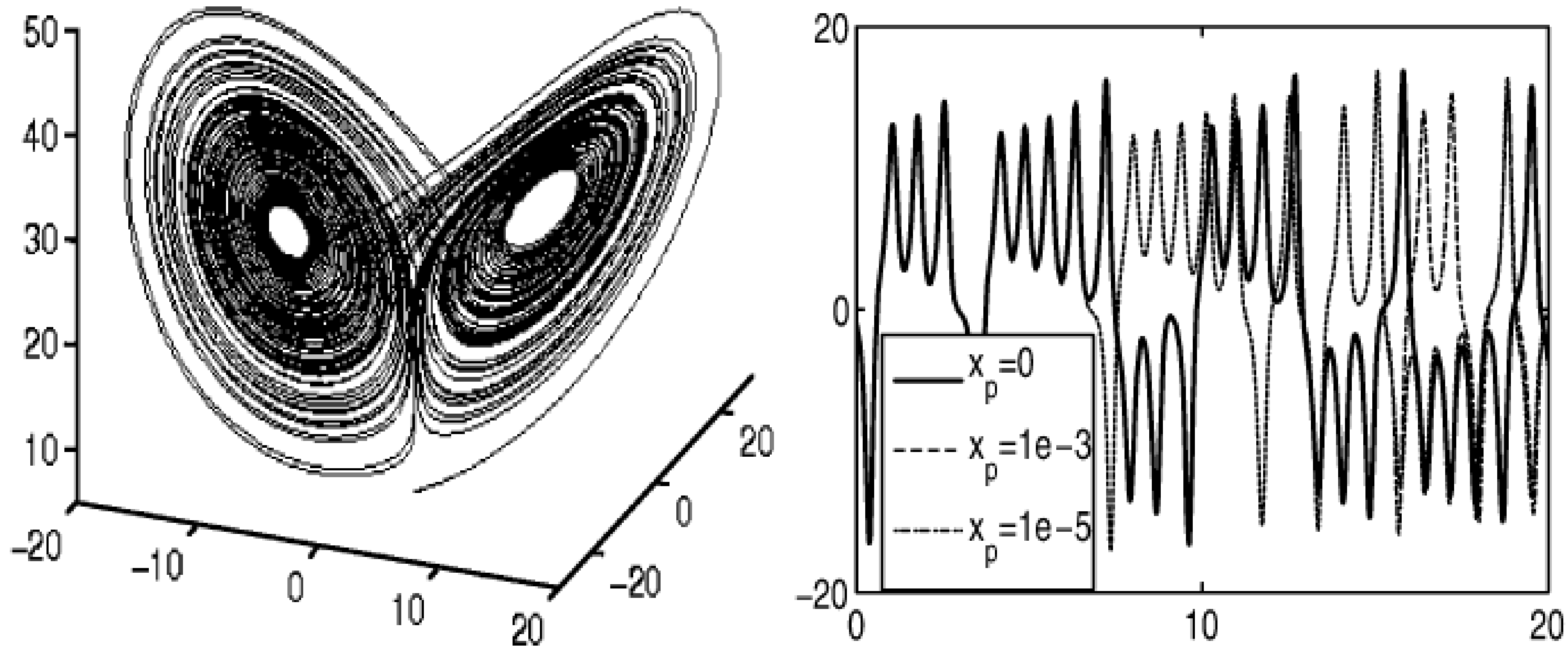


Figure 1.2. Left: Illustration of the attractor of the Lorenz system $\dot{x} = \sigma(y - x)$, $\dot{y} = \rho x - y - xz$, $\dot{z} = -\beta z + xy$ by one orbit in 3D phase space, $\sigma = 10$, $\beta = 8/3$, $\rho = 27$. Right: $x(t)$ for three nearby initial conditions, i.e., $x_1(0)$, $x_2(0) = x_1(0) + 10^{-3}$, and $x_3(0) = x_1(0) + 10^{-5}$, $y(0), z(0)$ always the same. The orbits behave completely different after a certain time, i.e., the orbits to $x_2(0)$ and $x_3(0)$ deviate from the unperturbed one after $t \approx 7$ and $t \approx 16$, respectively. It can be shown that the prediction time goes logarithmically with the precision of the initial conditions.

Moreover, **computers are fast, but never fast enough.** A three-dimensional body $[0,1]^3$ discretized with 100 points in each direction leads to 10^6 variables. A discretization in 1000 points in each direction yields 10^9 variables. Therefore, due to practical reasons one has to decide before what quantities shall be computed. Then the scheme can be adapted to the computation of these quantities.

We are especially interested in problems which cannot be directly studied numerically, i.e., where first **analysis is needed to reduce the dimensionality of the problem.** This is for instance the case in so called spatially extended domains, which means that the wave length of typical solutions is much smaller than the size of the underlying physical domain. In this case often the modeling over an unbounded domain is more reasonable. Then, via a multiple scaling perturbation ansatz simpler models can be derived to describe the phenomena under consideration. These models, called **modulation equations**, belong to the best studied nonlinear PDEs with a status in some scientific areas similar to the three classical linear PDEs from above. Besides the study of these basic nonlinear PDEs from a dynamical systems point of view, one of our main objectives will be the connections between these models and real world problems by going beyond the formal derivation of these modulation equations. This will be called the **justification** of the reduced models.

Example 1.3.1. The digital transport of information in glass fibers is done by sending 0s and 1s through the fiber. In most modern technologies the physical realization of a 1 is an electromagnetic pulsemodulating a carrier wave with a wave length of a few hundred nanometers. There are a number of relevant questions related to the transport of information. For instance:

- Which form is optimal for a pulse to travel a long distance?
- Which distance do two pulses initially need in order to stay separated during the complete journey through the fiber?
- How many kilometers can a pulse travel without an amplifier?
- How do pulses interact if the carrier waves have different frequencies?

There is dispersion in the fiber and thus in general the energy concentrated in a pulse will spread. Moreover, the fiber behaves nonlinearly. Hence, the answers to the above questions are not obvious at all. Numerical simulations, if possible, are much cheaper than experiments. However, suppose that the length of the fiber is 100km $= 10^5$m. Then, due to the wave length of light of approximately 10^{-7}m, a spatial discretization of Maxwell's equations in the fiber gives at least about 10^{12} points, still neglecting all three-dimensional effects. This number is too big for a direct numerical simulation.

A modulation equation helps. By perturbation analysis the **Nonlinear Schrödinger (NLS) equation**

$$\partial_\tau A = \mathrm{i}\nu_1 \partial_\xi^2 A + \mathrm{i}\nu_2 A|A|^2,$$

with $A(\xi,\tau) \in \mathbb{C}$, $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$ and coefficients $\nu_1, \nu_2 \in \mathbb{R}$, can be derived, describing the evolution of the envelope A of the pulse alone. On the relevant time scale the dynamics of the envelope of the pulse and the carrier wave which behaves linearly can approximately be separated. The properties of the original system, e.g., the refractive index of the material, and the underlying wave, condense to the coefficients $\nu_j \in \mathbb{R}$. The NLS equation is a **universal** modulation equation which describes slow modulations in time and space of the envelope of a spatially and temporarily oscillating underlying carrier wave in nonlinear dispersive equations.

The spatial discretization can thus be reduced from 10^{12} points to approximately 10^5 or less points, which is quite manageable for numerical schemes. Moreover, a number of problems can be solved analytically for the NLS equation, which is a so called **completely integrable system**. In particular, if $\nu_1\nu_2 > 0$ it has explicit so called **soliton solutions**. These solitons give the optimal form of pulses for the transport of information. These questions will be discussed in detail in Chapter 11. $\rfloor$

Example 1.3.2. At the end of the 20th century a new generation of high speed ferries has caused serious problems, especially those that cross the Channel between England and France and those operating in the Marlborough sound in New Zealand. The waves created by these ferries can propagate without loss of energy over large distances, and thus retain the potential to create enormous havoc when they come ashore, and as a consequence of a fatal accident and other damage there are now speed limits for these ferries [**Ham99**].

Again a modulation equation gives an idea to understand these phenomena. The **Korteweg-de Vries (KdV) equation**

$$\partial_\tau A = \nu_1 \partial_\xi^3 A + \nu_2 A \partial_\xi A,$$

with $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$, $A(\xi, \tau) \in \mathbb{R}$ and coefficients $\nu_j \in \mathbb{R}$ can be derived with the help of a perturbation ansatz. The KdV equation is a **universal** modulation equation which describes long waves of small amplitude, where the original system condenses to the coefficients $\nu_j \in \mathbb{R}$.

Like the NLS equation, this famous nonlinear equation possesses soliton solutions, very robust solitary waves. These waves interact like particles, i.e., after some nonlinear interaction they reform and look exactly as before the interaction. This observation, made in the middle of the 1960s, that solutions of a PDE show simultaneously the behavior of a particle and a wave, had a big influence on nonlinear science due to the similarity with the particle-wave dualism in quantum mechanics.

For a long time the KdV equation has also been suggested as a model for the description of tsunamis, water waves of only a few meters height, but with a length of up to 100km, i.e., in the ocean they cannot be observed by eye. In the 5000m deep pacific ocean they move with a very high velocity of around 700km/h. If they approach land they become slower and steeper, and cause serious floodings. However, data which is now available from the tsunami at Christmas 2004 in the Indian Ocean seem to indicate that soliton dynamics had played at least for this tsunami no role on the open sea. The validity of the KdV equation will be discussed in Chapter 12. ⌋

Example 1.3.3. Since the 1960s, systems near the onset of a finite wave length instability have been analyzed in detail using modulation equations. These amplitude modulations describe slow changes in time and space of the envelope of the finite wave length pattern close to the first instability. The most famous and generic of such equations is the **Ginzburg-Landau (GL) equation**

$$\partial_\tau A = \nu_2 \partial_\xi^2 A + \nu_0 A + \nu_3 A |A|^2,$$

with $\tau \geq 0$, $\xi \in \mathbb{R}$, $A(\xi, \tau) \in \mathbb{C}$ and coefficients $\nu_j \in \mathbb{C}$. Famous **pattern forming systems** which can be described with the GL equation are reaction-diffusion systems such as the Brusselator, and hydrodynamical stability problems, such as the Couette-Taylor problem, Bénard's problem, or Poiseuille flow. A big part of Part IV, namely Chapter 10, is devoted to the justification of this so called GL approximation for various classes of systems. We explain and prove that the difference of true solutions of the pattern forming systems and the associated GL approximations remains small on the natural time scale of this approximation, and thus prove rigorously that the GL equation makes correct predictions about the dynamics of the original pattern forming systems.

Instead of "modulation equation", in particular the GL equation in the above context is also called the "amplitude equation". Although derived differently, the GL model also plays a crucial role in superconductivity.

A phenomenological model for pattern formation close to the first instability of a spatially homogeneous solution is the **Swift-Hohenberg** equation [**SH77**]

$$\partial_t u = -(1+\partial_x^2)^2 u + \alpha u - u^3, \tag{1.4}$$

with $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$ and control parameter $\alpha \in \mathbb{R}$. This fourth order scalar PDE is probably the simplest example to apply the "Ginzburg-Landau formalism". For small $\alpha =: \varepsilon^2 > 0$, plugging the ansatz

$$u(x,t) = \varepsilon A(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.} \tag{1.5}$$

into (1.4) and sorting w.r.t. powers of ε yields the GL equation

$$\partial_T A = 4\partial_X^2 A + A - 3|A|^2 A \tag{1.6}$$

at order ε^3. ⌋

As already said, the mathematical analysis of the approximation by these three 'generic' modulation equations, namely the KdV, the NLS, and the GL equation, will be one of the mathematical objectives of Part IV of this book. Beside these 'generic' equations there are many more.

Example 1.3.4. The Burgers equation

$$\partial_t u = \partial_x^2 u - \partial_x(u^2),$$

with $t \geq 0$, $x \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$ arises for instance as a modulation equation for small amplitude long waves on the surface of a viscous liquid running down an inclined plane. It describes this system in case when the trivial solution, the so called Nusselt solution, which possesses a parabolic flow profile and a flat top surface, is spectrally stable. This is the case when the inclination angle θ, which serves as a control parameter in this physical problem, is below a critical value θ_c. This model is used for instance for flood forecasts in rivers.

If the inclination angle is increased, the Nusselt solution becomes unstable via a so called sideband instability. Above the threshold of instability the Kuramoto-Shivashinsky-perturbed KdV equation serves as modulation equation. After some rescaling it has the form

$$\partial_t u = -\partial_x^3 u - \frac{1}{2}\partial_x(u^2) - \varepsilon(\partial_x^2 + \partial_x^4)u,$$

with $t \geq 0$, $x \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, and where $0 < \varepsilon \approx \sqrt{\theta - \theta_c} \ll 1$ is a small parameter. Therefore, complicated dynamics that are present in this equation occur directly at the first instability of the inclined plane problem. The dynamics is dominated by traveling pulse trains consisting of unstable pulses. Time series of the position of the pulses indicate the occurrence of chaotic dynamics. This situation is relevant for cooling units. Again the 3D Navier-Stokes equations for the water flowing down the unit is replaced by a simpler model still containing very complicated dynamics.

Another situation where the Burgers equation arises as a modulation equations are phase or wave number modulations of stable periodic pattern in a pattern forming system, while phase (or wave number) modulations of unstable pattern are generically described by Kuramoto-Shivashinsky type of equations. ⌋

In summary, modulation equations are simpler PDEs, which can be derived by perturbation analysis, and which serve as models for more complicated systems. Hence, modulation equations are a part of mathematical modeling. In Part IV of this book, the derivation and the approximation properties of the above equations will be explained. We will analyze the original system with the help of the modulation equations. We will explain to which extent conclusions based on the modulation equations can be proven to be correct. We will show how mathematics can decide which model of all possible proposed models is the right one. We will explain that modulation equations are universal models, i.e., exactly the same modulation equation describes the same phenomena in completely different physical systems. The much simpler modulation equations itself will be analyzed in Part III of this book.

1.4. Overview

In order to keep the book as an introductory text and as self-contained as possible, in Part I we explain basic dynamical systems concepts for ODEs, such as phase space, fixed points, periodic solutions, attractors, stability and instability, bifurcations and amplitude equations.

In Part II we start to transfer the dynamical systems concepts from finite to infinite dimensions. There are major differences due to the non-equivalence of norms in infinite-dimensional vector spaces and the loss of compactness of closed bounded sets. We explain that PDEs over bounded domains can be considered as dynamical systems with countably many degrees of freedom. As applications we discuss the Chafee-Infante problem and the Navier-Stokes equations.

We have already explained in the previous subsection our choice of equations for the Parts III and IV. In Part III we consider basic model PDEs posed on the real line, namely the Kolmogorov-Petrovsky-Piscounov (KPP) or Fisher equation, the Burgers equation, the Nonlinear Schrödinger (NLS) equation, the Korteweg-deVries (KdV) equation, and the Ginzburg-Landau (GL) equation. We explain fundamental PDE phenomena as diffusion, dispersion, and transport, discuss local and global existence and uniqueness, and construct stationary solutions, or traveling front and pulse solutions, using ODE techniques from Part I. We also give some first results for attractors on unbounded domains and a brief introduction to reaction-diffusion systems.

Part IV is devoted to the analysis of the more complicated systems with the help of the scalar model equations from Part III, which now reappear as modulation equations. Additionally we explain useful concepts such as diffusive stability and spatial dynamics.

At the end of each chapter we collect a number of **exercises**. We in general do not claim any originality for them, and many are taken from the literature, though in some cases we cannot trace back our source. As usual, the exercises are a crucial part of this book.

1.4.1. Grasshopper's Guide. To some extent the four parts of this book are intended to be independent. Moreover, the chapters are kept as self-contained as possible, such that the reader may start to read directly about his or her favorite equation. Therefore, we also give the following guide.

Part I can obviously be read independently of the rest of the book. It is an example-oriented basic course on finite-dimensional dynamical systems which together with Chapters 5 and 6 (and possibly Chapter 13) yields a two semester course about finite- and infinite-dimensional dynamical systems. Chapters 7 and 8 of Part III can subsequently serve as a basis for a seminar.

An alternative one or two semester course is given by §2.2-§2.3 about basic nonlinear ODE dynamics combined with (parts or all of) Part III and some parts of Part IV, for instance the beginning of Chapter 10. Other chapters of Part IV can then serve as a basis for a seminar.

There are other possibilities, for instance Chapter 3 about dissipative ODE dynamics combined with some dissipative PDE dynamics, chosen out of Chapters 5, 7, §8.3, Chapters 9-10, parts of Chapter 13, and Chapter 14. Similarly, Chapter 4 about conservative ODE dynamics could be combined with some conservative PDE dynamics, chosen for instance out of §8.1, §8.2, and Chapters 11 and 12. If the reader is familiar with the contents of Part I and Part II and is interested in an introduction to the mathematical theory of modulation equations, then we recommend to start reading in Part IV and going back to Part III where needed.

Nevertheless, the reader can also work through the book from the beginning to the end.

1.4.2. Recommended literature. Good classical books about PDEs are [**CH89, Joh91, Eva98, Sal08, Vas15**], while [**Str92, SVZZ13, Olv14, Log15a**] give more elementary introductions to PDEs. Books which look at PDEs from a dynamical systems point of view are [**Hen81, Tem97, RR04, Rob01**]. These books cover and extend material similar to that in the first two parts of our book, in particular Part II, while for instance [**SS99b, KP13**] discuss in more detail parts of what is treated in our Part III. For a general background on the functional analytic methods in our book we recommend [**Alt16, Wer00**], but the needed material can be found in most books on functional analysis. For more physically oriented introductions to PDEs see [**Fow97, BK00, TM05, Deb05**], for an overview of developments in the theory of PDEs in the 20th century see [**BB98**], and for an encyclopedic work on PDEs see [**Tay96**]. A "visual approach" to PDE with many motivating pictures is [**Mar07**]. For ODEs we refer for instance to [**Chi06, HSD04, Tes12, Log15b**]. Beginning in Part II, at the end of most Chapters we give an outlook and hints for further reading.

1.4.3. Software. There are many software packages for the numerical solution of ODEs and the graphical presentation of solutions. `Matlab`, `Maple`, and `Mathematica` have built in facilities, and there are various simple to use Java applets available. We strongly encourage the reader to do own experiments with any of these programs.

From the above remarks about the very different types of PDEs it readily follows that there cannot be a general tool for all types of PDEs. However, tools for specific types of PDEs, both commercial and free are widely available. We use some short self-written `matlab` scripts to illustrate some PDE dynamics, mostly for model problems. However, we do not discuss any numerical methods behind these programs and refer to [**Uec09**] and the references therein. For the computation of so called bifurcation diagrams we refer to `AUTO` [**Doe07, Dea16**] and `pde2path` [**UWR14**].

Exercises

1.1. Classify the following PDEs as linear or nonlinear.

a) $\partial_t u = \sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}(a^{ij}u) + \sum_{i=1}^{d} \partial_{x_i}(b^i u)$, $a^{ij}, b^i : \mathbb{R}^d \to \mathbb{R}$ smooth functions.

b) $i\partial_t u = \Delta u$. c) $\partial_t V = rV - rS\partial_S V - \frac{1}{2}\sigma^2 S^2 \partial_S^2 V$, $(r, \sigma \in \mathbb{R})$.

d) $\partial_t^2 u = -\partial_x^4 u$. e) $\partial_t u = \Delta(u^\gamma)$, $(\gamma > 0)$.

f) $\partial_t u = \mathrm{div} F(u)$, $F : \mathbb{R} \to \mathbb{R}^d$ a smooth function. g) $\partial_t u = \partial_x^3 u + u\partial_x u$.

1.2. Constant coefficient second order linear partial differential equations in $\mathbb{R}^2$ can be written as

$$Lu = -\sum_{i,j=1,2} a^{ij}\partial_{x_i}\partial_{x_j} u + \sum_{i=1,2} b^i \partial_{x_i} u + c = 0.$$

The operator L is called elliptic if the eigenvalues of the symmetric matrix $A = (a^{ij})$ are strictly positive. It is called hyperbolic if they are nonzero, but have different signs. It is called parabolic if the associated quadratic form $(\partial_x \to \xi,\ \partial_y \to \eta)$ defines a parabola. Classify

a) $3\partial_x^2 u + 10\partial_x\partial_y u + 15\partial_y^2 u + 36\partial_x u + 12\partial_y u + 17 = 0$;

b) $3\partial_x^2 u + 4\partial_x u + \partial_y u + 2 = 0$.

1.3. Consider the PDE $\partial_t u = \partial_x u$ for $u = u(x, t)$.

a) Find the general solution for $x \in \mathbb{R}$.

b) Solve the PDE for $x \in (0, 1)$ with the initial condition $u(x, 0) = 1$ for $x \in (0, 1)$ under the boundary condition $u(1, t) = \cos t$.

c) Is it possible to solve the PDE for $x \in (0, 1)$ with the initial condition $u(x, 0) = 1$ for $x \in (0, 1)$ and the boundary condition $u(0, t) = \cos t$?

1.4. Consider a membrane $\Omega = (0, 1)^2$ which is fixed at the boundary $\partial\Omega$, i.e., $u|_{\partial\Omega} = 0$.

a) Make an ansatz $u(x, y, t) = v(t)\sin(m\pi x)\sin(n\pi y)$, $(n, m \in \mathbb{N})$ for the solutions of $\partial_t^2 u = \Delta u$. Which equation is satisfied by v?

b) Solve the equation for v with the initial conditions $v(0) = 0$ and $\dot{v}(0) = 1$.

c) Sketch for fixed $m, n \in \mathbb{N}$ the set of $(x, y) \in \Omega$, for which $u(x, y, t) = 0$ for all $t \in \mathbb{R}$.

Chapter 2

Basic ODE dynamics

The first part of this book is about nonlinear dynamics in $\mathbb{R}^d$. It consists of this chapter, Chapter 3 about dissipative dynamics, and Chapter 4 about Hamiltonian dynamics. In this part we provide some basic concepts of nonlinear dynamics. In order to avoid the various functional analytic difficulties associated with PDEs we restrict to the finite-dimensional situation, i.e., we consider ODEs

$$\dot{u}(t) = f(u(t), t),$$

with $u(t) \in \mathbb{R}^d$, $f : \mathbb{R}^d \times I \to \mathbb{R}^d$ a continuous vector field which is locally Lipschitz-continuous w.r.t. its first argument, where $I \subset \mathbb{R}$ is an open interval, usually $I = \mathbb{R}$, and where $\dot{u}(t)$ denotes the derivative of the function u w.r.t. time t. In general it is not possible to obtain explicit solutions, and so our main goal is to provide tools for the understanding of the qualitative behavior of solutions.

Some notation. The initial value problem consists in finding a solution of the ODE to an initial value u_0 given at an initial time $t_0 \in I$. A solution of the initial value problem is a function $u \in C^1(I_0, \mathbb{R}^d)$ which fulfills the ODE, where $I_0 \subset I$ is again an open interval, $t_0 \in I_0$, and $u(t_0) = u_0$. This solution is denoted by $u(t, t_0, u_0)$. If f on the right-hand side of the ODE does not depend explicitly on time, i.e., $f = f(u)$, then the ODE is called autonomous, and we may assume $t_0 = 0$ and write $u(t, u_0)$ for the solution of the initial value problem.

Absolutely fundamental for the understanding of nonlinear dynamics is the understanding of the dynamics of linear systems which we therefore consider first. Then we introduce basic concepts of nonlinear dynamics. These are the local and global existence and uniqueness of solutions, special solutions such as fixed points, periodic solutions, homoclinic and heteroclinic orbits, and further concepts such as stability and instability, invariant manifolds, ω-limit-sets, attractors, and chaotic dynamics. Many of these concepts will later be transferred to nonlinear PDEs. Moreover, the search for special solutions, such as front or pulse solutions for the PDEs, in later chapters very often lead to ODE problems as they are considered here.

The behavior and the analysis of an ODE or of a PDE strongly differ between dissipative and conservative systems. In Chapter 3 we provide the strategy and the tools to tackle dissipative systems. Such systems are typically characterized by the existence of compact absorbing sets, i.e., compact sets into which all solutions finally enter. In dissipative systems very often more complicated and eventually chaotic dynamics occur through bifurcations if some external parameter is varied. After introducing a number of elementary bifurcations for one- and two-dimensional systems we introduce with the Lyapunov-Schmidt reduction and the center manifold theorem two reduction methods which allow us to find these elementary bifurcations in higher dimensional systems, too. Chapter 3 is closed by presenting some routes of bifurcations to chaotic behavior in dissipative systems.

The systems considered in Chapter 3 change the volume in phase space, but many systems in nature preserve the volume in phase space, especially those of classical mechanics. Thus, Chapter 4 is devoted to Hamiltonian ODE dynamics. We provide some tools for their analysis and explain basic facts about their behavior, which shows fundamental differences compared to that of the systems of Chapter 3. For instance, a globally attracting fixed point cannot exist in conservative or volume-preserving systems. Therefore, stability and instability proofs or the route to chaotic behavior must be completely different. The starting point of the bifurcation analysis is not a globally attracting fixed point, but a so called completely integrable system. In Chapter 4 we also discuss KAM theory which allows to understand the behavior of systems which are small perturbations of completely integrable Hamiltonian systems.

The ideas presented in this first part will reappear in subsequent sections. For instance, Chapter 3 about dissipative ODE dynamics contains basic tools which will be used in Chapters 5-7, §8.3-Chapter 10, and Chapter 14 about dissipative PDE dynamics. Similarly, Chapter 4 about conservative ODE dynamics contains basic tools which will help to understand §8.1, §8.2, and Chapters 11-12 about conservative PDE dynamics.

We emphazise that the purpose of Part I is not to give a comprehensive overview about ODEs. Rather we present the basic ideas of nonlinear dynamics as needed in subsequent parts of this book in the analysis of PDEs.

There are a number of excellent textbooks on nonlinear ODE dynamics, many of them also reviewing basic linear ODE dynamics. An elementary and very readable account on ODE dynamics and bifurcations is [**HK91**], a very applied and example oriented approach is used in [**Str94**], and an excellent modern presentation is given in [**Tes12**]. Alternatives and complements to these textbooks are for instance [**Chi06, Ver96, Rob04a, Rob04b**]. More advanced texts include [**GH83, KH97, Wig03, HSD04**]. In [**SH96, Lyn04**] discrete dynamical systems and ODEs are treated from a numerical point of view, and [**Dev89**] focusses on discrete chaotical dynamical systems. For the bifurcation aspects of ODEs, and in particular center manifolds, we again refer to [**GH83, Wig03**], and to [**Kuz04, Erm02**] for invariant manifolds from a numerical point of view. Our favorite books on Hamiltonian systems and KAM theory are [**Arn78, Thi88, MH92**], see also [**Way96**].

2.1. Linear systems

Fundamental for the understanding of nonlinear dynamics is the understanding of linear dynamics. Linear ODEs occur for instance as linearizations around fixed points or periodic solutions. The solution of these linear ODEs and the variation of constant formula, which allow us to solve inhomogeneous linear problems, will be the basis for stability proofs for fixed points and periodic solutions of nonlinear ODEs in §2.3. Moreover, this technique will be generalized to semi-linear dissipative PDEs and a number of conservative PDEs in Parts II-IV for proving the local existence and uniqueness of solutions of PDEs.

A linear ODE is an equation

$$\dot{u}(t) = A(t)u(t) + g(t) \tag{2.1}$$

for an unknown function $u \in C^1(I, \mathbb{R}^d)$, where $I \subset \mathbb{R}$ is an interval, $A(t) \in \mathbb{R}^{d\times d}$ is a $d\times d$-matrix with entries $a_{ij}(t)$, and where $g(t) \in \mathbb{R}^d$ is an inhomogeneity. We generally think of t as time, and for simplicity we assume that A and g are at least continuous w.r.t. t. Together with an initial condition $u|_{t=t_0} = u_0 \in \mathbb{R}^d$ we have an initial value problem. Equation (2.1) is called homogeneous, if $g(t) \equiv 0$, i.e., if

$$\dot{u}(t) = A(t)u(t), \tag{2.2}$$

and (2.1), respectively (2.2), are called autonomous if A and g in (2.1), respectively A in (2.2), do not depend on t.

It is well-known that the initial value problems for (2.1) and (2.2) have unique solutions, and that the solutions of (2.1) form a d-dimensional affine space and the solutions of (2.2) a d-dimensional vector space. This will briefly be recalled in §2.1.2. We restrict ourselves to those parts of the theory which are needed in subsequent sections. That is, we restrict to A independent of time, and to A periodic in time, i.e., $A(t) = A(t+T)$ for a $T > 0$. Other important classes of linear systems are asymptotically constant systems, i.e., $A(t) \to A_\pm$ for $t \to \pm\infty$. They appear as linearization around so called homoclinic or heteroclinic orbits.

2.1.1. Notation. Let X be a real or complex vector space. A map $\|\cdot\| : X \to \mathbb{R}$ is called norm, if for all $u, v \in X$ and $\lambda \in \mathbb{R}$, respectively $\lambda \in \mathbb{C}$,

(i) $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$,

(ii) $\|\lambda u\| = |\lambda|\ \|u\|$,

(iii) $\|u + v\| \leq \|u\| + \|v\|$.

In $\mathbb{R}^d$, the major examples are the ℓ_1-norm $\|u\|_1 = \sum_{j=1}^d |u_j|$, the Euclidean or ℓ_2-norm $\|u\|_2 = (u^T u)^{1/2} = (\sum_{j=1}^d |u_j|^2)^{1/2}$, and the ℓ_∞- or maximum-norm $\|u\|_\infty = \max_{j=1,\dots,d} |u_j|$.

Concepts such as convergence in $\mathbb{R}^d$, or later on stability and instability for ODEs in $\mathbb{R}^d$, are independent of the chosen norm in $\mathbb{R}^d$. The reason for this is the equivalence of norms in finite-dimensional vector spaces.

Theorem 2.1.1. *All norms in $\mathbb{R}^d$ are equivalent, i.e., for two norms $\|\cdot\|$ and $\|\cdot\|_*$ there exist positive constants C_1, C_2 such that for all $u \in \mathbb{R}^d$ we have*

$$\|u\| \leq C_1 \|u\|_* \leq C_2 \|u\|.$$

Proof. Obviously, it is sufficient to establish the estimates between an arbitrary norm $\|\cdot\|$ and the $\|\cdot\|_\infty$-norm. By the triangle inequality we have

$$\|u\| = \left\| \sum_{j=1}^d u_j e_j \right\| \leq \sum_{j=1}^d |u_j| \|e_j\| \leq \left(\sum_{j=1}^d \|e_j\| \right) \|u\|_\infty .$$

For the second estimate let $M = \{u \in \mathbb{R}^d : \|u\|_\infty = 1\}$. Then $f : M \to \mathbb{R}$, $u \mapsto \|u\|^{-1}$ is a continuous map by definition. Suppose that f is unbounded on M, i.e., there exists a sequence $(u^n)_{n \in \mathbb{N}}$ with $\|u^n\| \to 0$ for $n \to \infty$. Since the finitely many coordinates u_j^n satisfy $|u_j^n| \leq 1$ there exists a convergent subsequence $u^{n_k} \to \xi \in M$ for $k \to \infty$. By the continuity of the norm we have $\|u^{n_k}\| \to \|\xi\| = 0$ for $k \to \infty$ which implies $\xi = 0$ contradicting $\xi \in M$. Hence, there exists a $C > 0$ such that $\sup_{u \in M} \|f(u)\| = C < \infty$ and so $\|u\| \geq \frac{1}{C}$ for all $u \in M$ which finally leads to $\|u\| \geq \frac{1}{C} \|u\|_\infty$. $\square$

Remark 2.1.2. The reason why we gave a proof of this well known theorem is that in infinite dimensions there are infinitely many non-equivalent norms. This has a number of consequences for the subsequent analysis of PDEs. It is possible that uniqueness but no global existence of solutions is known in one space, and global existence but no uniqueness of solutions is known in another space, but there is no space known where both properties hold simultaneously. So far this is exactly the state of the art for one of the Millennium problems of the Clay Foundation, namely the global existence and uniqueness of smooth solutions of the 3D Navier-Stokes equations, which will be discussed in Chapter 6. ⌋

The $d \times d$-matrices form a normed vector space of dimension d^2. With the matrix multiplication AB they form an algebra. An important matrix norm is given by the operator norm

$$\|A\|^* = \sup\left\{\frac{\|Au\|}{\|u\|} : u \in \mathbb{R}^d \setminus \{0\}\right\}.$$

Obviously $\|Au\| \leq \|A\|^* \, \|u\|$, and so from

$$\|ABu\| \leq \|A\|^* \, \|Bu\| \leq \|A\|^* \, \|B\|^* \, \|u\|$$

it follows

$$\|AB\|^* \leq \|A\|^* \, \|B\|^*,$$

i.e., the matrices form a Banach algebra w.r.t. matrix multiplication and operator norm. For arbitrary norms the matrix norm and the vector norm are called compatible if

$$\|Au\| \leq \|A\| \, \|u\|.$$

Examples of such compatible norms are

$$\begin{aligned}
\|\cdot\| &= \|\cdot\|_1, \qquad \|A\|^* &&= \sup_{k=1,\ldots,d} \sum_{j=1}^d |a_{jk}|, \\
\|\cdot\| &= \|\cdot\|_2, \qquad \|A\|^* &&= \left(\sum_{i,j=1}^d |a_{jk}|^2\right)^{1/2}, \\
\|\cdot\| &= \|\cdot\|_\infty, \qquad \|A\|^* &&= \sup_{j=1,\ldots,d} \sum_{k=1}^d |a_{jk}|.
\end{aligned}$$

In the following we always use compatible norms, and $\mathbb{R}^d$ will be equipped with the Euclidean norm, if not indicated otherwise. Finally, we remark that for all norms

$$\left\|\int_{t_0}^t u(\tau)\,\mathrm{d}\tau\right\| \leq \int_{t_0}^t \|u(\tau)\|\,\mathrm{d}\tau.$$

2.1.2. Local existence and uniqueness. We briefly recall that the initial value problem for (2.2) has a unique solution, and that the solutions of (2.2) form a d-dimensional vector space. The following local existence and uniqueness result and many other results in this book are based on the contraction mapping principle which is absolutely fundamental in nonlinear analysis. We recall that a metric space M is called complete, if every Cauchy sequence in M possesses a limit in M.

Theorem 2.1.3. (Contraction mapping principle or fixed point theorem of Banach) *Let (M, d) be a complete metric space and $F : M \to M$ a contraction, i.e., there exists a $\kappa \in (0, 1)$ such that*

$$d(F(x), F(y)) \leq \kappa d(x, y)$$

for all $x, y \in M$. Then F has a unique fixed point $x^* \in M$, i.e., $x^* = F(x^*)$.

Proof. We first prove the uniqueness. Suppose that there exist two different fixed points x^* and y^*. Then

$$d(x^*, y^*) = d(F(x^*), F(y^*)) \leq \kappa d(x^*, y^*).$$

Since $\kappa \in (0, 1)$, it follows that $d(x^*, y^*) = 0$ and hence $x^* = y^*$, in contradiction to the assumption.

We define the sequence $x_{n+1} = F(x_n)$ with $x_0 \in M$ arbitrary, but fixed. Then for $m \geq n$

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq \sum_{j=n}^{m-1} \kappa^j d(x_1, x_0) \leq \frac{\kappa^n}{1-\kappa} d(x_1, x_0).$$

Hence, for all $\varepsilon > 0$ there exists an $N > 0$, such that for all $n, m > N$:

$$d(x_m, x_n) \leq \frac{\kappa^N}{1-\kappa} d(x_1, x_0) \leq \varepsilon,$$

i.e., $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence. Since M is complete, there exists an $x^* \in M$, such that $x^* = \lim_{n\to\infty} x_n$.

The limit x^* is a fixed point due to the continuity of F, i.e.,

$$F(x^*) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = x^*. \qquad \square$$

Corollary 2.1.4. *Let $(X, \|\cdot\|)$ be a Banach space, M be a closed subset of X, and $F : M \to M$ be a contraction. Then F has a unique fixed point $x^* \in M$.*

Our first version of the local existence and uniqueness of solutions for (2.2) is as follows.

Lemma 2.1.5. *Consider (2.2) with initial condition $u|_{t=t_0} = u_0$ and continuous $A = A(t)$. Then there exists a $\delta > 0$ independent of u_0 such that (2.2) has a unique solution $u \in C^1((t_0 - \delta, t_0 + \delta), \mathbb{R}^d)$ satisfying $u|_{t=t_0} = u_0$.*

Proof. The proof is based on the application of the contraction mapping theorem to the integrated ODE

$$u(t) = u_0 + \int_{t_0}^{t} A(s)u(s)\,\mathrm{d}s =: F(u)(t),$$

where $F : M \to M$ with $M = C^0([t_0 - \delta, t_0 + \delta], \mathbb{R}^d)$. Fix a $T_0 > 0$ and define $C_0 = \sup_{t\in[t_0-T_0,t_0+T_0]} \|A(t)\|$ which is finite due to the continuity of $t \mapsto A(t)$. Then we have $\|F(u) - F(v)\|_M \le \delta C_0 \|u - v\|_M$, where $\|u\|_M = \sup_{t\in[t_0-\delta,t_0+\delta]} \|u(t)\|_{\mathbb{R}^d}$, such that F is a contraction for instance for $\delta = \min\{1/(2C_0), T_0\}$. From $u \in M$ it follows that $F(u) \in C^1((t_0-\delta, t_0+\delta), \mathbb{R}^d)$. Since u is a fixed point we also have $u = F(u) \in C^1((t_0 - \delta, t_0 + \delta), \mathbb{R}^d)$. $\square$

Remark 2.1.6. By the last argument, it is easy to see that the m-times differentiability of $t \mapsto A(t)$ implies inductively the m+1-times differentiability of $t \mapsto u(t)$. $\rfloor$

Lemma 2.1.5 only asserts local existence and uniqueness. The way to show existence and uniqueness of solutions beyond $t_0 + \delta$ is to prove bounds on $u(t_0 + \delta)$. The key tool is Gronwall's inequality which will be used in many proofs below. We first restrict to a simple version [**Ver96**, Theorem 1.2].

Lemma 2.1.7. (Gronwall's inequality) *For $t \in (t_0, t_0 + a)$ with $a > 0$, and ϕ and ψ non-negative continuous functions assume that*

$$\phi(t) \le \int_{t_0}^{t} \psi(s)\phi(s)\,\mathrm{d}s + \delta. \tag{2.3}$$

Then for all $t \in (t_0, t_0 + a)$ we have

$$\phi(t) \le \delta \mathrm{e}^{\int_{t_0}^{t} \psi(s)\,\mathrm{d}s}.$$

Proof. Dividing (2.3) by its right-hand side and multiplication of both sides with $\psi(t)$ yields after integration that

$$\int_{t_0}^{t} \frac{\psi(\tau)\phi(\tau)}{\int_{t_0}^{\tau} \psi(s)\phi(s)\,\mathrm{d}s + \delta}\,\mathrm{d}\tau \le \int_{t_0}^{t} \psi(\tau)\,\mathrm{d}\tau$$

which implies that $\ln\left(\int_{t_0}^{t} \psi(s)\phi(s)\,\mathrm{d}s + \delta\right) - \ln\delta \le \int_{t_0}^{t} \psi(\tau)\,\mathrm{d}\tau$ and finally that

$$\int_{t_0}^{t} \psi(s)\phi(s)\,\mathrm{d}s + \delta \le \delta \exp\left(\int_{t_0}^{t} \psi(\tau)\,\mathrm{d}\tau\right).$$

By assumption $\phi(t)$ is smaller than the expression on the left-hand side. $\square$

From the integrated ODE

$$u(t) = u_0 + \int_{t_0}^{t} A(s)u(s)\,\mathrm{d}s$$

we find the inequality

$$\|u(t)\|_{\mathbb{R}^d} \le \|u_0\|_{\mathbb{R}^d} + \left|\int_{t_0}^{t} \|A(s)\|\ \|u(s)\|_{\mathbb{R}^d}\,\mathrm{d}s\right|$$

and so by Gronwall's inequality

$$\|u(t)\|_{\mathbb{R}^d} \leq \|u_0\|_{\mathbb{R}^d} \exp\left(\left|\int_{t_0}^{t} \|A(s)\| \,\mathrm{d}s\right|\right). \tag{2.4}$$

Since continuous functions stay bounded on compact intervals, from (2.4) it follows that for continuous $t \mapsto A(t)$ the solutions $t \mapsto u(t)$ exist for all $t \in \mathbb{R}$.

Theorem 2.1.8. *Consider* (2.2) *with initial condition* $u|_{t=t_0}=u_0$ *and a continuous* $A=A(t)$ *for all* $t \in \mathbb{R}$. *Then there exists a unique solution* $u \in C^1(\mathbb{R}, \mathbb{R}^d)$ *satisfying* $u|_{t=t_0} = u_0$.

Proof. We choose an arbitrary, but fixed $T_0 > 0$. We apply Lemma 2.1.5 with the initial condition $u|_{t=t_0} = u_0$, which gives a unique solution on $[t_0, t_0 + \delta)$ and $u|_{t=t_0+\delta/2} = u_1$. Inductively we apply Lemma 2.1.5 now with the initial condition $u|_{t=t_0+n\delta/2} = u_n$, which gives a unique solution on $[t_0 + n\delta/2, t_0 + n\delta/2 + \delta)$ and $u|_{t=t_0+(n+1)\delta/2} = u_{n+1}$. Doing this until $(n+1)\delta/2 \geq T_0$ gives us the solution for all $t \in [t_0, t_0 + T_0)$. Proceeding similarly for negative $t - t_0$ gives the solution for all $t \in (t_0 - T_0, t_0 + T_0)$. Since $T_0 > 0$ was arbitrary we are done. □

Lemma 2.1.9. *The solutions of* (2.2) *form a d-dimensional vector space.*

Proof. Since the solutions of (2.2) depend linearly and one-to-one on the initial conditions u_0 we have that the set of solutions of (2.2) is isomorphic via $u \mapsto u(t_0)$ to the space of initial conditions, i.e., $\mathbb{R}^d$. □

Definition 2.1.10. *The matrix-valued function* $t \mapsto \phi(t)$ *is called fundamental matrix if* $\dot{\phi}(t) = A(t)\phi(t)$ *and if* $\phi(t_0)$ *has full rank for a* $t_0 \in \mathbb{R}$.

Remark 2.1.11. From the local existence and uniqueness Theorem 2.1.8, it immediately follows that $\phi(t)$ has full rank for all $t \in \mathbb{R}$. If $t \mapsto \phi(t)$ is a fundamental matrix then $u(t) = \phi(t)\phi(t_0)^{-1}u_0$ solves the initial value problem with $u|_{t=t_0} = u_0$. If $t \mapsto \psi(t)$ is another fundamental matrix then $\phi(t)\phi(t_0)^{-1} = \psi(t)\psi(t_0)^{-1}$ such that there exists an invertible matrix $C = \phi(t_0)^{-1}\psi(t_0)$ which is independent of t with $\psi(t) = \phi(t)C$. ⌋

2.1.3. The variation of constant formula. Associated to (2.2) we define the linear solution operator $S(t,s) : \mathbb{R}^d \mapsto \mathbb{R}^d$ through

$$S(t,s)u_0 = u(t,s,u_0),$$

where $u(t,s,u_0)$ is the solution of (2.2) with initial value $u|_{t=s} = u_0$. For fixed $s,t \in \mathbb{R}$ the linear map $S(t,s) : \mathbb{R}^d \to \mathbb{R}^d$ is one-to-one, i.e., an invertible matrix, with $S(t,s)^{-1} = S(s,t)$. The solution of the inhomogeneous problem (2.1), i.e.,

$$\dot{u}(t) = A(t)u(t) + g(t) \tag{2.5}$$

can be expressed in terms of the inhomogeneity $g = g(t)$ and the solution operator $S(t,s)$. Differentiation of $u(t) = S(t,s)y(t)$ w.r.t. t and using (2.5) shows that

$$\dot{u}(t) = \partial_t S(t,s)y(t) + S(t,s)\dot{y}(t) = A(t)S(t,s)y(t) + g(t).$$

Since $S(t,s)$ solves the homogeneous problem $\partial_t S(t,s) = A(t)S(t,s)$ we obtain

$$\dot{y}(t) = S(t,s)^{-1}g(t) = S(s,t)g(t).$$

Integration yields $y(t) - y(s) = \int_s^t \dot{y}(\tau)\,\mathrm{d}\tau = \int_s^t S(s,\tau)g(\tau)\,\mathrm{d}\tau$ and therefore

$$\begin{aligned} u(t) =& S(t,s)y(t) = S(t,s)y(s) + S(t,s)\int_s^t S(s,\tau)g(\tau)\,\mathrm{d}\tau \\ =& S(t,s)u(s) + \int_s^t S(t,\tau)g(\tau)\,\mathrm{d}\tau, \end{aligned}$$

since $S(s,s) = I$. This formula is called the variation of constant formula. It will be used in stability proofs and in proofs of the local existence and uniqueness of solutions of nonlinear problems. In case $S(t,s) = \mathrm{e}^{(t-s)A}$, see the subsequent §2.1.4, the variation of constant formula specializes to

$$u(t) = \mathrm{e}^{tA}u(0) + \int_0^t \mathrm{e}^{(t-\tau)A}g(\tau)\,\mathrm{d}\tau. \tag{2.6}$$

2.1.4. The exponential matrix. In general, (2.2) can only be solved explicitly in case $d = 1$. For $d \geq 2$, if a solution is known of the d-dimensional problem, then the dimension of the problem can be reduced by one, i.e., after the reduction a linear ODE in $d-1$ space dimensions has to be solved, cf. [**Cod61**, Page 118]. However, in case

$$\dot{u} = Au, \quad u|_{t=0} = u_0, \tag{2.7}$$

with $A \in \mathbb{R}^{d\times d}$ independent of t, all solutions can be computed explicitly. The ansatz $u(t) = \mathrm{e}^{\lambda t}\widehat{u}$ yields the eigenvalue problem $A\widehat{u} = \lambda\widehat{u}$, and in case that A has d linearly independent eigenvectors $\phi_1, \ldots, \phi_d \in \mathbb{R}^d$ with eigenvalues $\lambda_1, \ldots, \lambda_d$, the general solution reads $u(t) = \sum_{i=1}^d c_i \mathrm{e}^{\lambda_i t}\phi_i$ with $c_i, \ldots, c_d \in \mathbb{R}$. In case that there are complex eigenvalues or Jordan blocks this formula becomes slightly more complicated. Equation (2.7) appears as linearization around fixed points of nonlinear systems and hence plays a crucial role.

Remark 2.1.12. Linear scalar equations of n^{th} order

$$y^{(n)}(t) + \sum_{j=0}^{n-1} a_j y^{(j)}(t) = 0,$$

with $a_j \in \mathbb{R}$, can always be written as a linear first order system. For the construction of the explicit solution this is not necessary. The r different zeroes λ_k with multiplicity k_r, i.e. $\sum_{k=1}^{r} k_r = n$, of the characteristic equation $\lambda^n + \sum_{j=0}^{n-1} a_j \lambda^j = 0$ allow to construct the general solution, namely

$$y(t) = \sum_{k=1}^{r} \sum_{j=0}^{k_r - 1} c_{k,j} \mathrm{e}^{\lambda_k t} t^j$$

with n coefficients $c_{k,j}$. See [**Log15b**, Chapter 2]. ⌋

From a theoretical point of view, the following representation formula turns out to be useful. The solution of (2.7) is given by

$$u(t) = \mathrm{e}^{tA} u_0 = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} u_0. \tag{2.8}$$

We have absolute and uniform convergence w.r.t. t on every compact interval due to

$$\sum_{n=0}^{\infty} \left\| \frac{(tA)^n}{n!} u_0 \right\|_{\mathbb{R}^d} \leq \sum_{n=0}^{\infty} \frac{\|A\|^n t^n}{n!} \|u_0\|_{\mathbb{R}^d} \leq \mathrm{e}^{\|A\| t} \|u_0\|_{\mathbb{R}^d}.$$

Obviously this also holds for the series differentiated w.r.t. t. Similarly, we obtain the estimate

$$\|\mathrm{e}^{tA} u_0\|_{\mathbb{R}^d} \leq \mathrm{e}^{t\|A\|} \|u_0\|_{\mathbb{R}^d}.$$

Moreover, $\mathrm{e}^{tA} u_0$ solves (2.7) due to

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!} u_0 \right) = \sum_{n=1}^{\infty} \frac{A(tA)^{n-1}}{(n-1)!} u_0 = A \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} u_0,$$

where the time derivative and the infinite sums can be interchanged due to the uniform convergence w.r.t. t on compact intervals. Moreover, A and the infinite sum can be interchanged due to the boundedness and hence continuity of A.

The solution operator e^{tA} can be expressed in terms of the Jordan normal form J of A. The change of coordinates $u = Sy$ in $\dot{u} = Au$ yields $\dot{y} = S^{-1} A S y = J y$. We have

$$\mathrm{e}^{tA} u_0 = S \mathrm{e}^{tJ} y_0 = S \mathrm{e}^{tJ} S^{-1} u_0, \tag{2.9}$$

or equivalently

$$\begin{aligned} S^{-1}\mathrm{e}^{tA}S =& S^{-1}\left(1 + tA + \frac{t^2A^2}{2} + \dots\right)S \\ =& \left(1 + tS^{-1}AS + \frac{t^2S^{-1}ASS^{-1}AS}{2} + \dots\right) \\ =& \left(1 + tJ + \frac{t^2J^2}{2} + \dots\right) = \mathrm{e}^{tJ}. \end{aligned}$$

Hence, it is sufficient to consider e^{tJ} for J a matrix in Jordan normal form, i.e.,

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_r \end{pmatrix} \text{ with } J_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

Since

$$\exp(tJ) = \exp\left(\begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{pmatrix} t\right) = \begin{pmatrix} \mathrm{e}^{tJ_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{tJ_r} \end{pmatrix}$$

it is sufficient to consider

$$\mathrm{e}^{tJ_j} = \mathrm{e}^{t(\lambda_j I + N_k)} = \mathrm{e}^{\lambda_j t I}\mathrm{e}^{tN_k}$$

due to $IN_k = N_k I$ with the $k \times k$-matrix

$$N_k = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Hence, it remains to compute

$$\mathrm{e}^{tN_k} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} N_k^\nu.$$

We have

$$\begin{aligned} N_k^\mu &= (\delta_{i,j-\mu}), & \text{for } \mu &= 0, ..., k-1, \\ N_k^\mu &= 0, & \text{for } \mu &= k, k+1, ... \end{aligned}$$

and so finally

$$\mathrm{e}^{tN_k} = \begin{pmatrix} 1 & t & \dots & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & \ddots & & \frac{t^{k-2}}{(k-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 1 & t \\ 0 & & \dots & 0 & 1 \end{pmatrix}.$$

This representation formula immediately yields a statement about the stability of the fixed point $u = 0$ of the ODE $\dot{u} = Au$.

Definition 2.1.13. *a) The fixed point $u = 0$ is called asymptotically stable for $\dot{u} = Au$ if for all $u_0 \in \mathbb{R}^d$ we have $u(t) = \mathrm{e}^{tA}u_0 \to 0$ for $t \to \infty$.*

b) The fixed point $u = 0$ is called stable for $\dot{u} = Au$ if for all $u_0 \in \mathbb{R}^d$ we have that $u(t) = \mathrm{e}^{tA}u_0$ stays bounded for all $t \geq 0$.

c) In all other cases the origin $u = 0$ is called unstable.

Theorem 2.1.14. *a) If all eigenvalues of A have strictly negative real parts, then $u = 0$ is asymptotically stable.*

b) If A possesses no eigenvalue with positive real part and if all eigenvalues with real part zero possess the same algebraic and geometric multiplicity, i.e., no non-trivial Jordan block, then the origin $u = 0$ is stable.

c) In all other cases, i.e., if A possesses at least one eigenvalue with strictly positive real part or at least one eigenvalue with real part zero with algebraic multiplicity bigger than the geometric multiplicity, then the origin $u = 0$ is unstable.

2.1.5. Linear planar systems. The behavior of the solutions of two-dimensional autonomous systems $\dot{u} = f(u)$ can be visualized with the help of so called phase portraits. As a first step we discuss and visualize the behavior of linear two-dimensional autonomous systems

$$\dot{u} = Au, \quad u(t) \in \mathbb{R}^2, \quad A \in \mathbb{R}^{2\times 2}. \tag{2.10}$$

In order to visualize the behavior of (2.10) we have a number of possibilities, which we shall later also apply to nonlinear systems. In doing so we also classify the different kinds of fixed points.

1) We plot the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$. This turns out to be not that helpful due to the in general strongly varying length of f.
2) Therefore, we plot in most cases the direction field $\alpha f/\|f\|_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ for a fixed $\alpha > 0$, i.e., in every point $u \in (h\mathbb{Z})^2$ of a grid with width h we plot a vector of $\mathbb{R}^2$ of fixed length α.

3) We plot the flow, i.e., a number of chosen orbits. These are curves which are defined by the solutions $t \mapsto u(t) \in \mathbb{R}^2$.

A combination of 2) and 3) is called a phase portrait. The choice of the size of h and of the orbits depends on the problem, and also is a matter of taste.

Remark 2.1.15. The vector field and the direction field are both tangent vectors of the solution $t \mapsto u(t)$. Hence, the differential equations $\dot{u} = f(u)$ and $\dot{u} = f(u)/\|f(u)\|_{\mathbb{R}^2}$ have the same orbits, i.e., solution curves in the phase plane, although their dynamics are very different. ⌋

One more option is to plot the nullclines. These are the sets

$$N_j := \{(u_1, u_2) \in \mathbb{R}^2 : f_j(u_1, u_2) = 0\},$$

for $j = 1, 2$, where the vector field is vertical, respectively horizontal. The intersection points of N_1 and N_2 give the fixed points u^* of the ODE, i.e., points with $f(u^*) = 0$. If the solution starts in a fixed point, the solution stays in that fixed point, i.e., $u(t) = u^*$ for all t. Often, nullclines at least partially coincide with coordinate axis, and, moreover, for (non-degenerate) linear systems we have $N_1 \cap N_2 = \{(0, 0)\}$ as the only fixed point.

Due to §2.1.4 it is sufficient to consider (2.2) with $A \in \mathbb{R}^{2\times 2}$ in Jordan normal form. There are the following cases.

a) The eigenvalues have the same algebraic and geometric multiplicity, i.e.,
$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with **a1)** $0 \neq \lambda_j \in \mathbb{R}$ or **a2)** $0 \neq \lambda_1 = \overline{\lambda_2}$. In case a1) we distinguish three subcases i) $\lambda_1 = \lambda_2$, ii) $\lambda_1 > \lambda_2 > 0$ and iii) $\lambda_1 > 0 > \lambda_2$. All other cases are obtained from i)-iii) by a reversal of time $t \mapsto -t$.

b) The eigenvalue has geometric multiplicity one and algebraic multiplicity two, i.e., $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is a Jordan block with $\lambda \in \mathbb{R}$.

c) The degenerate case of at least one eigenvalue $\lambda = 0$. Besides the trivial case $A = 0$ there are the cases **c1)** $0 = \lambda_1 < \lambda_2$ and **c2)** $\lambda = 0$ with geometric multiplicity one and algebraic multiplicity two.

In the following we consider a number of examples to visualize these cases, see Figure 2.1.5.

a1 i): Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e., $\dot{u}_1 = u_1$, $\dot{u}_2 = u_2$. For the solutions we find $u_1(t) = \mathrm{e}^t u_1(0)$, $u_2(t) = \mathrm{e}^t u_2(0)$. The orbits are straight lines since $u_1(t)/u_2(t) = u_1(0)/u_2(0) = const.$. The nullclines are

$N_1 = \{(0,y) : y \in \mathbb{R}\}$ and $N_2 = \{(x,0) : x \in \mathbb{R}\}$. The fixed point $u^* = (0,0)$ is called a source or unstable node. If time is reversed, i.e., if $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $u^* = (0,0)$ is called a sink or stable node.

a1 ii): Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, i.e., $\dot{u}_1 = 2u_1$, $\dot{u}_2 = u_2$. For the solutions we find $u_1(t) = \mathrm{e}^{2t}u_1(0)$, $u_2(t) = \mathrm{e}^{t}u_2(0)$. The orbits are parabolas since $u_1(t)/(u_2(t))^2 = u_1(0)/(u_2(0))^2 = const.$. The phase portrait is robust w.r.t. small perturbations, i.e., $A = \mathrm{diag}(1.9, 1.1)$ has a similar phase portrait. Again the fixed point $u^* = (0,0)$ is called a source.

a1 iii): Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e., $\dot{u}_1 = u_1$, $\dot{u}_2 = -u_2$. For the solutions we find $u_1(t) = \mathrm{e}^{t}u_1(0)$, $u_2(t) = \mathrm{e}^{-t}u_2(0)$. The orbits are hyperbolas since $u_1(t)u_2(t) = u_1(0)u_2(0) = const.$. The phase portrait is robust w.r.t. small perturbations, i.e., $A = \mathrm{diag}(1.1, -0.9)$ gives a similar phase portrait. The fixed point $u^* = (0,0)$ is called a saddle.

a2 i): Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e., $\dot{u}_1 = u_2$, $\dot{u}_2 = -u_1$. Then $\widetilde{u} = u_1 + \mathrm{i}u_2$ solves $\dot{\widetilde{u}} = -\mathrm{i}\widetilde{u}$. The solution $\widetilde{u}(t) = \mathrm{e}^{-\mathrm{i}t}\widetilde{u}(0)$ leaves the circles $|\widetilde{u}(t)|^2 = u_1^2(t) + u_2^2(t) = |\widetilde{u}(0)|^2 = const.$ invariant. Hence, the orbits are circles, and $u^* = (0,0)$ is called a center. The phase portrait is not robust w.r.t. small perturbations. In general after the perturbation we obtain the phase portrait from a2 ii). However, in applications often additional effects such as a conserved quantity enforce the robustness of centers w.r.t. the class of possible perturbations. See for instance Chapter 4.

a2 ii): Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. For $\widetilde{u} = u_1 + \mathrm{i}u_2$ we obtain the equation $\dot{\widetilde{u}} = (1 - \mathrm{i})\widetilde{u}$ which is solved by $\widetilde{u}(t) = \mathrm{e}^{t}\mathrm{e}^{-\mathrm{i}t}\widetilde{u}(0)$. In polar coordinates $\widetilde{u}(t) = r(t)\mathrm{e}^{\mathrm{i}\phi(t)}$ with $r(t) \in \mathbb{R}$ and $\phi(t) \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ we obtain $\dot{r} = r$ and $\dot{\phi} = -1$ with solution $r(t) = \mathrm{e}^{t}r(0)$ and $\phi(t) = (\phi(0) - t) \bmod 2\pi$. The orbits are spirals. The phase portrait is robust w.r.t. small perturbations. Here $u^* = (0,0)$ is called an unstable vortex or spiral.

b): Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, i.e., $\dot{u}_1 = u_1 + u_2$, $\dot{u}_2 = u_2$. For the second equation we obtain $u_2(t) = \mathrm{e}^{t}u_2(0)$. The variation of constant formula applied to the first equation yields

$$u_1(t) = \mathrm{e}^t u_1(0) + \int_0^t \mathrm{e}^{t-s}\mathrm{e}^s u_2(0)\,\mathrm{d}s = \mathrm{e}^t u_1(0) + \mathrm{e}^t t u_2(0).$$

Here $u^* = (0,0)$ is called a degenerate node. The phase portrait is not generic, since a Jordan block only occurs with probability zero under all matrices. However, it can make sense to keep the Jordan block as starting point of the analysis.

c1): Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The general solution is $u_1(t) = \mathrm{e}^t u_1(0)$, $u_2(t) = u_2(0) + \mathrm{e}^t u_1(0)$, and there is the line of fixed points $u_1 = 0$.

c2): Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The general solution is $u_1(t) = u_1(0) + u_2(0)t$, $u_2(t) = u_2(0)$, i.e., the flow is parallel to the line of fixed points $u_2 = 0$.

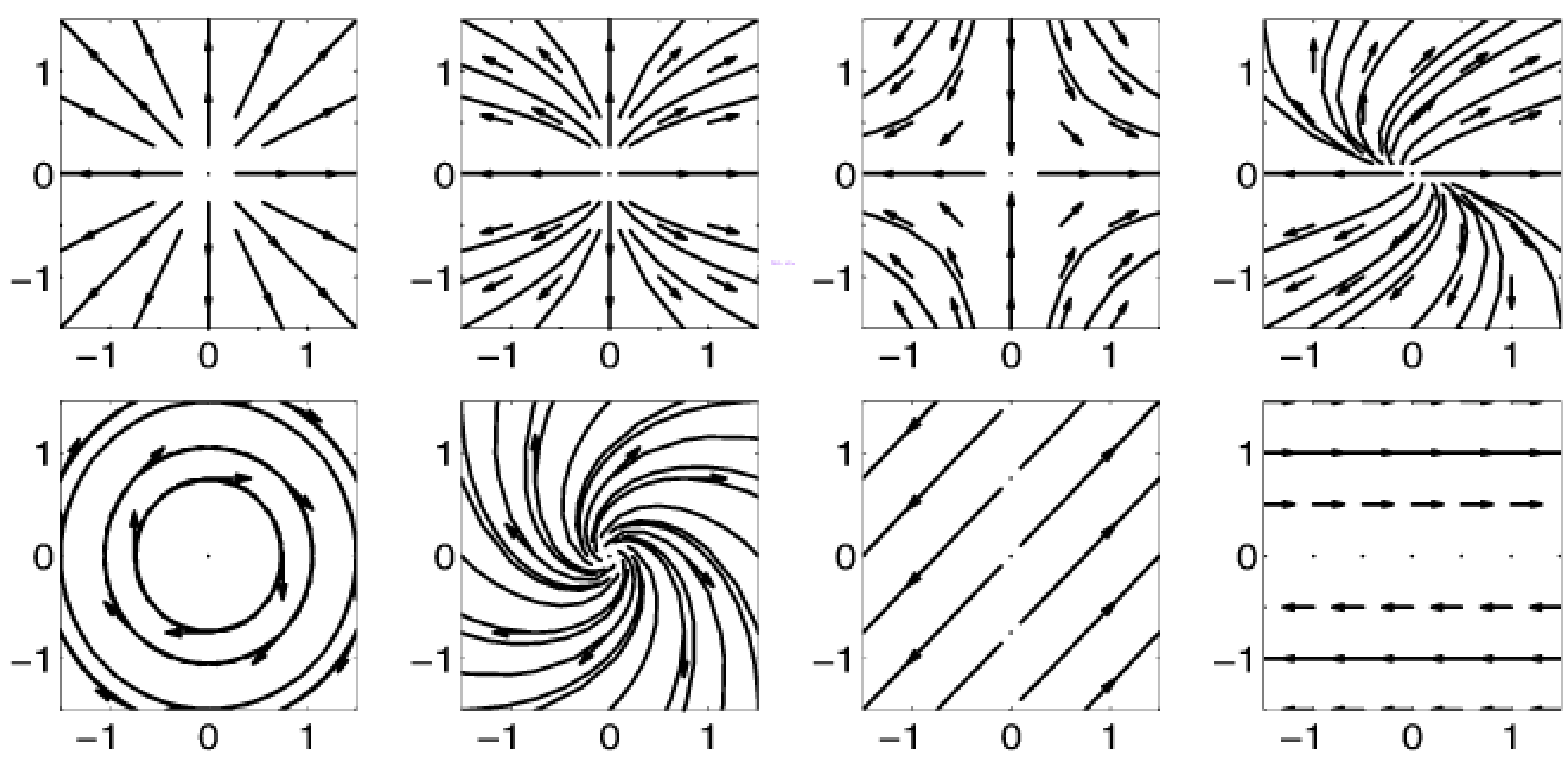


Figure 2.1. Phase portraits for a1i), a1ii), a1iii), and b) in the first row, for a2i), a2ii), c1) and c2) in the second row.

2.1.6. Linear systems with periodic coefficients. Equations

$$\dot{u}(t) = A(t), u(t) \quad \text{with} \quad A(t) = A(t+T), \tag{2.11}$$

for a fixed $T > 0$, appear for instance as linearizations around time-periodic solutions of nonlinear systems. Hence, they will play an important role in the following. In contrast to the case of t-independent matrices A, where an arbitrary shift of time still gives the same system, in case of time-periodic $A = A(t)$ only integer multiples of the basic period T can be allowed.

Lemma 2.1.16. *With* $u(t, t_0, u_0)$ *also* $u(t + nT, t_0 + nT, u_0)$ *with* $n \in \mathbb{N}$ *solves* (2.11).

The following theorem is fundamental.

Theorem 2.1.17. ***(Floquet)****Each fundamental matrix $\phi(t)$ can be written as a product*

$$\phi(t) = P(t)\mathrm{e}^{tB}$$

of two $d \times d$-matrices, with $P(t) = P(t+T)$ and B a constant $d \times d$-matrix.

Proof. Since

$$\dot{\phi}(t+T) = A(t+T)\phi(t+T) = A(t)\phi(t+T)$$

with $\phi(t)$ also $\phi(t+T)$ is a fundamental matrix. Hence, there exists an invertible $d \times d$-matrix C, such that

$$\phi(t+T) = \phi(t)C.$$

Each invertible $d \times d$-matrix C can be written as $C = \mathrm{e}^{TB}$ with B a non-unique $d \times d$-matrix. As example consider $C = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_j > 0$. Then a logarithm is given by $B = \mathrm{diag}(\ln \lambda_1, \ldots, \ln \lambda_d)$. For the general case use $-1 = \mathrm{e}^{\mathrm{i}\pi}$ and the expansion of $\ln(1+\lambda)$ in case of Jordan blocks. See Exercise 2.5. For $P(t) = \phi(t)\mathrm{e}^{-tB}$ we obtain

$$P(t+T) = \phi(t+T)\mathrm{e}^{-(t+T)B} = \phi(t)C\mathrm{e}^{-TB}\mathrm{e}^{-tB} = \phi(t)\mathrm{e}^{-\mathrm{tB}} = \mathrm{P(t)}. \quad \square$$

Definition 2.1.18. *The matrix $C = \mathrm{e}^{TB}$ is called monodromy matrix. The eigenvalues of C are called Floquet multipliers, and the eigenvalues of B are called Floquet exponents.*

Floquet exponents from a different matrix B differ only by adding integer multiples of $2\pi\mathrm{i}/T$. The Floquet multipliers are unique. Suppose that two fundamental matrices $\phi = \phi(t)$ and $\psi = \psi(t)$ are given. By Remark 2.1.11 then $\psi^{-1}(t)\phi(t) = S$ is independent of time and hence

$$C_\phi = \phi(t)^{-1}\phi(t+T) = S^{-1}\psi(t)^{-1}\psi(t+T)S = S^{-1}C_\psi S.$$

As a consequence the matrices C_ϕ and C_ψ have the same eigenvalues. The T-periodic transformation $u(t) = P(t)y(t)$ gives

$$P(t)\dot{y}(t) + \dot{P}(t)y(t) = \dot{u}(t) = A(t)u(t) = A(t)P(t)y(t)$$

and thus

$$\begin{aligned}
\dot{y}(t) =& P(t)^{-1}(A(t)P(t) - \dot{P}(t))y(t) \\
=& P(t)^{-1}(A(t)P(t) - \dot{\phi}(t)\mathrm{e}^{-tB} - \phi(t)(-B)\mathrm{e}^{-tB})y(t) \\
=& P(t)^{-1}(A(t)P(t) - A(t)\phi(t)\mathrm{e}^{-tB} + \phi(t)\mathrm{e}^{-tB}B)y(t) \\
=& P(t)^{-1}(A(t)P(t) - A(t)P(t) + P(t)B)y(t) = By(t).
\end{aligned}$$

For the stability of $u = 0$, it is therefore sufficient to consider B. If all eigenvalues λ of B satisfy $\mathrm{Re}\,\lambda < 0$, we have the asymptotic stability of $u = 0$, see Theorem 2.1.19.

Alternatively, by Lemma 2.1.16 for all $n \in \mathbb{N}$ and $\tau \in [0, T)$, we have

$$\begin{aligned} u(t, 0, u_0) =& u(nT + \tau, 0, u_0) = u(nT + \tau, nT, u(nT, 0, u_0)) \\ =& u(nT + \tau, nT, u(nT, (n-1)T, u((n-1)T, 0, u_0))) \\ =& u(\tau, 0, u(T, 0, u(T, 0, \ldots, u(T, 0, u_0)) \ldots))) \\ =& \Phi_\tau \circ \Phi_T \circ \ldots \circ \Phi_T(u_0), \end{aligned}$$

where $\Phi_t u_0 = u(t, 0, u_0)$. Since $\tau \in [0, T)$ for the long-time dynamics only the iteration of the time T-map

$$\Phi_T = \phi(T)\phi(0)^{-1} = \left(P(T)\mathrm{e}^{TB}\right)\left(P(0)\mathrm{e}^{0B}\right)^{-1} = P(0)C_\phi P(0)^{-1},$$

is of interest, where we used $P(T) = P(0)$ and $\mathrm{e}^{0B} = I$. The proof of the following theorem is an easy exercise.

Theorem 2.1.19. *In a discrete dynamical system $u_{n+1} = Cu_n$ we have:*

a) If all eigenvalues μ of C satisfy the condition $|\mu| < 1$, then $u = 0$ is asymptotically stable, i.e., $\lim_{n\to\infty} u(n, u_0) = 0$.

b) If C has an eigenvalue μ with $|\mu| > 1$, or a non-trivial Jordan block to an eigenvalue with $|\mu| = 1$, then $u = 0$ is unstable.

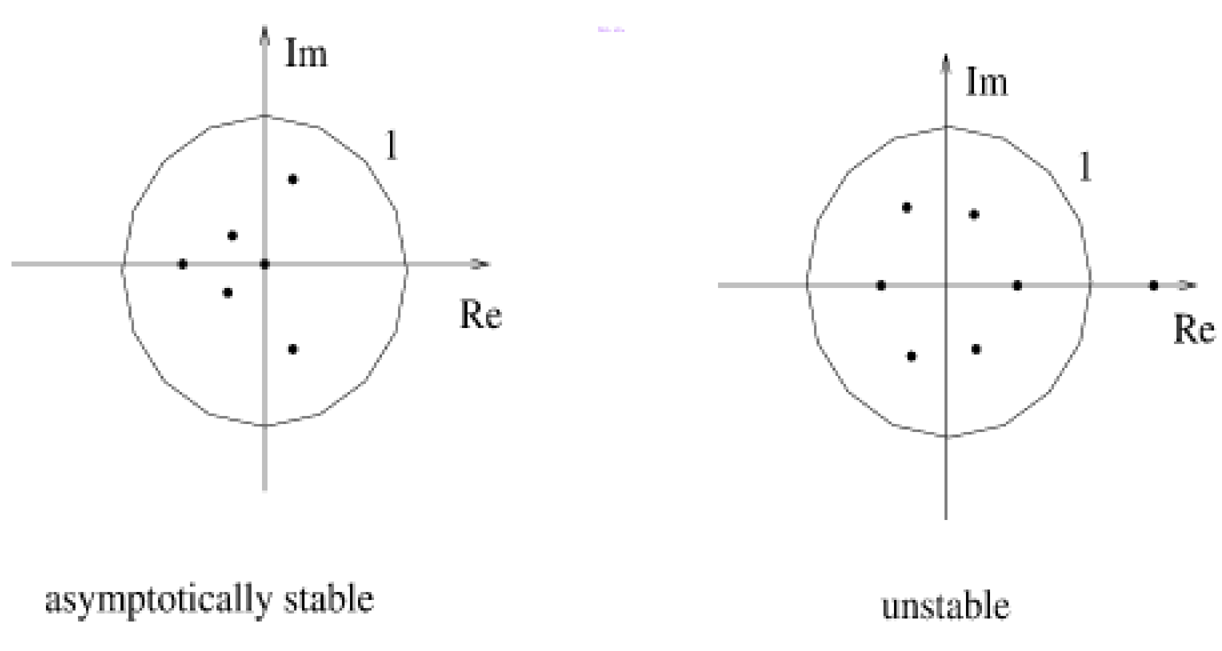


Figure 2.2. The eigenvalues of C in cases a) and b) of Theorem 2.1.19.

Example 2.1.20. Consider the iteration $x_{n+1} = Cx_n$, with $C = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$. The solution can be computed explicitly by the transformation $x = Sy$, with $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We find $x_{n+1} = SB^{n+1}S^{-1}x_0$, with $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. ⌋

Example 2.1.21. Consider the 1-periodic ODE $\dot{u}(t) = \cos^2(2\pi t)u(t)$ for $u(t) \in \mathbb{R}$. Using $\cos^2(2\pi t) = 1/2 + \cos(4\pi t)/2$ the solution with initial condition $u(0) = u_0$ is given by

$$u(t, 0, u_0) = u_0 \exp\left(\frac{t}{2} + \frac{\sin 4\pi t}{8\pi}\right)$$

and therefore

$$P(t) = \exp\left(\frac{\sin 4\pi t}{8\pi}\right) \quad \text{and} \quad e^{tB} = \exp\left(\frac{t}{2}\right).$$

We find the Floquet multiplier $e^{1/2}$ and the Floquet exponent $1/2$. The time-one-map is given through $\Phi_1 u_0 = e^{1/2} u_0$. ⌋

The following example [**MY60**] shows that in the periodic case the eigenvalues of the matrix $A(t)$ have no significance for the stability of $u = 0$.

Example 2.1.22. Consider $\dot{u} = A(t)u$ with

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}.$$

The characteristic polynomial is given by

$$\begin{aligned} &\left(1 - \frac{3}{2}\cos^2 t + \lambda\right)\left(1 - \frac{3}{2}\sin^2 t + \lambda\right) - \left(1 - \frac{3}{2}\sin t \cos t\right)\left(1 + \frac{3}{2}\sin t \cos t\right) \\ &\quad = \lambda^2 + 2\lambda - \frac{3}{2}\left(\cos^2 t + \sin^2 t\right)\lambda + 1 - \frac{3}{2}\left(\cos^2 t + \sin^2 t\right) \\ &\qquad + \frac{9}{4}\cos^2 t \sin^2 t + 1 - \frac{9}{4}\cos^2 t \sin^2 t \\ &\quad = \lambda^2 + \frac{1}{2}\lambda + \frac{1}{2}, \end{aligned}$$

i.e., the eigenvalues are independent of t and are given by $\lambda_{1,2} = (-1 \pm i\sqrt{7})/4$. Therefore, we expect $u = 0$ to be stable, but there is the solution

$$\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} e^{t/2},$$

which is unbounded for $t \to \infty$. With the help of this solution the 2×2-system of ODEs can be reduced to a scalar equation which can be solved with the method of separation of variables. Hence, the Floquet exponents can be computed explicitly. They are given by $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -1$. ⌋

2.1.7. An outlook on amplitude equations. We close this review of linear ODE theory with a first glimpse at what will be one of the main subjects of this book, namely reduction methods and amplitude (and modulation) equations. Consider the weakly damped linear oscillator

$$\ddot{u} + 2\varepsilon\dot{u} + u = 0, \qquad u(0) = a, \quad \dot{u}(0) = 0, \tag{2.12}$$

with $u(t) \in \mathbb{R}$ and $0 < \varepsilon \ll 1$. The explicit solution is

$$u(t) = e^{-\varepsilon t}\left(a\cos(\omega t) + \frac{\varepsilon a}{\omega}\sin(\omega t)\right), \text{ where } \omega = \sqrt{1 - \varepsilon^2},$$

cf. Remark 2.1.12. However, we might also try an expansion w.r.t. ε, i.e., $u(t) = u_0(t) + \varepsilon u_1(t) + \mathcal{O}(\varepsilon^2)$. Plugging this ansatz into (2.12) and sorting w.r.t. powers in ε yields

$$\begin{aligned}
\mathcal{O}(\varepsilon^0): \quad & \ddot{u}_0(0) + u_0 = 0, \quad u_0(0) = a,\ \dot{u}_0(0) = 0 \\
& \Rightarrow u_0(t) = a\cos t, \\
\mathcal{O}(\varepsilon^1): \quad & \ddot{u}_1 + u_1 = 2a\sin t, \quad u_1(0) = 0,\ \dot{u}_1(0) = 0 \\
& \Rightarrow u_1(t) = -at\cos t + a\sin t,
\end{aligned}$$

and hence $u_{\mathrm{app}_1}(t) = a\cos t - \varepsilon ta\cos t + \varepsilon a\sin t + \mathcal{O}(\varepsilon^2)$. Comparing with u shows that the expansion only makes sense for $t = \mathcal{O}(1)$, and becomes completely useless after that.

With some physical insight, we may however directly see from (2.12) that $2\varepsilon\partial_t u$ corresponds to a weak damping, and hence we suspect that there are two time scales involved in (2.12). Thus we may try a multi-scale ansatz of the form

$$u(t) = A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega_0 t} + \text{c.c.}, \tag{2.13}$$

with $\omega_0 \in \mathbb{R}$ an a priori unknown (fast) frequency, and where $A = A(\tau) \in \mathbb{C}$ is a slowly varying (complex valued) amplitude. Then, e.g, $\frac{\mathrm{d}}{\mathrm{d}t}u = (\mathrm{i}\omega_0 + \varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau})A\mathrm{e}^{\mathrm{i}\omega_0 t} + \text{c.c.}$, and plugging into (2.12) we obtain

$$\begin{aligned}
\mathcal{O}(\varepsilon^0): \quad & -\omega_0^2 + 1 = 0, \quad A(0) = a/2 \qquad \Rightarrow \omega_0 = 1, \\
\mathcal{O}(\varepsilon^1): \quad & 0 = -2\mathrm{i}(\frac{\mathrm{d}}{\mathrm{d}\tau}A + A)\mathrm{e}^{\mathrm{i}t} + \text{c.c.}.
\end{aligned}$$

This yields $A(\tau) = \mathrm{e}^{-\tau}A(0)$, and thus

$$u_{\mathrm{app}_2}(t) = A(\tau)\mathrm{e}^{\mathrm{i}t} + \text{c.c.} + \mathcal{O}(\varepsilon) = a\mathrm{e}^{-\varepsilon t}\cos(t) + \mathcal{O}(\varepsilon),$$

which at least is a much better approximation of the true solution than u_{app_1}, see Fig. 2.3.

The equation $\frac{\mathrm{d}}{\mathrm{d}\tau}A = -A$ is called the amplitude equation for the ansatz (2.13) for the system (2.12), and here can be solved explicitly, like the original system. However, already in simple nonlinear ODEs in general neither the original equation nor the amplitude equation can be solved explicitly. Moreover, although the amplitude equation is usually a bit "simpler", this is not the essential characteristic. The main points are that the amplitude equation often falls into some universality class, and that it describes the system on long scales. Thus, if one has to use numerical methods, then the numerical costs are greatly reduced. For instance, in the present example we would reduce the numerical costs by a factor $1/\varepsilon$, e.g., by factor 10 if $\varepsilon = 0.1$. More drastic cost reductions may occur for PDEs, see Part IV of this book.

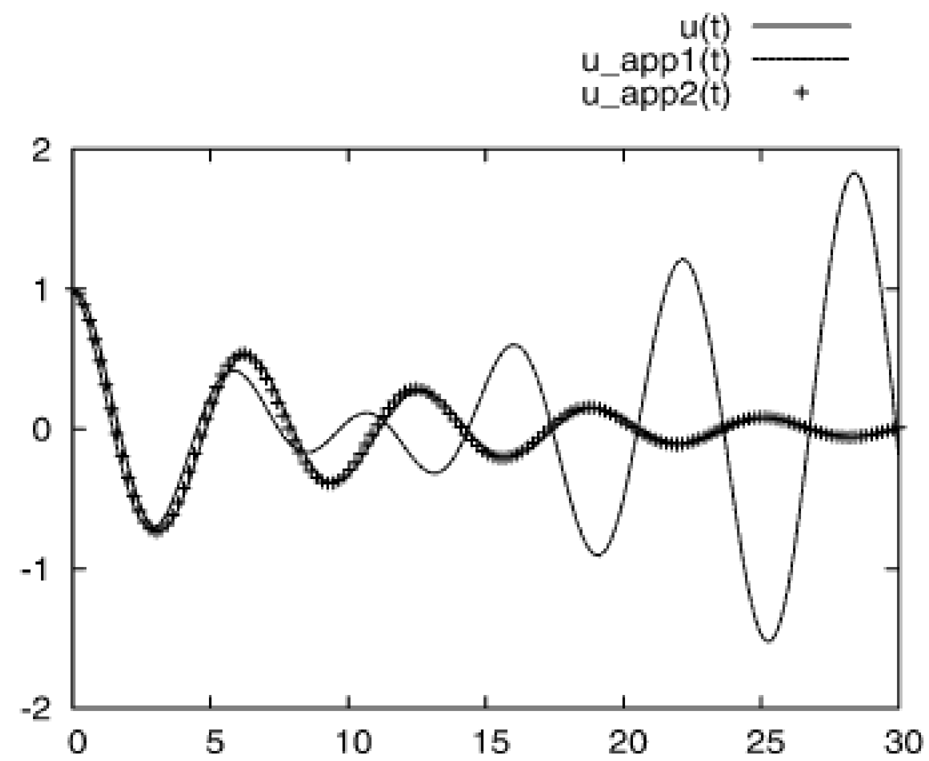


Figure 2.3. Exact solution and the two approximations for (2.12); $\varepsilon = 0.1$, $a = 1$.

2.2. Local existence and uniqueness for nonlinear systems

In this section we prove the local existence and uniqueness of solutions for nonlinear ODEs. We consider

$$\dot{u}(t) = f(u(t), t), \tag{2.14}$$

for an unknown function $u \in C^1(I, \mathbb{R}^d)$, where $I \subset \mathbb{R}$ is an interval and $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is called the vector field, which is called autonomous if it does not depend explicitly on time t. An initial value problem consists in finding solutions of (2.14) to the initial condition $u_0 \in \mathbb{R}^d$ at some time t_0, i.e., $u|_{t=t_0} = u_0$. If $d > 1$, then (2.14) is sometimes called a system of ODEs.

Sometimes f is not defined for all $t \in \mathbb{R}$ or for all $u \in \mathbb{R}^d$; the latter is for instance always the case if we consider a planar ODE in polar coordinates $u=(r, \phi)$ with hence $r \geq 0$. However, for a given initial condition u_0 at a time t_0 it is clear that for a local solution it is sufficient that f is only defined in a neighborhood of u_0 and t_0. The modifications needed in the theory below are obvious and thus for notational simplicity we generally assume that f is defined for all $u \in \mathbb{R}^d$ and all $t \in \mathbb{R}$.

For f locally Lipschitz-continuous w.r.t. u and continuous w.r.t. t we have the local existence and uniqueness of solutions. A function $f : \mathbb{R}^d \times I \to \mathbb{R}^d$ with $I \subset \mathbb{R}$ an open interval is called locally Lipschitz-continuous w.r.t. to the first variable if for all C_1 there exists a C_2 such that

$$\max\{\|u\|_{\mathbb{R}^d}, \|v\|_{\mathbb{R}^d}\} \leq C_1 \quad \Rightarrow \quad \sup_{t \in I} \|f(u,t) - f(v,t)\|_{\mathbb{R}^d} \leq C_2 \|u - v\|_{\mathbb{R}^d}.$$

Theorem 2.2.1. (Picard-Lindelöf) *Consider* (2.14) *with initial condition* $u|_{t=t_0} = u_0$ *and let* $f : \mathbb{R}^d \times I \to \mathbb{R}^d$ *be continuous w.r.t.* t *and locally Lipschitz-continuous w.r.t.* u. *For* $C_1 > 0$ *define* $M_0 = \{u \in \mathbb{R}^d : \|u -$

$u_0\|_{\mathbb{R}^d} \leq C_1\}$, $C_3 := \sup_{(u,t)\in M_0\times I} \|f(u,t)\|_{\mathbb{R}^d}$, *and denote the Lipschitz-constant in* M_0 *by* C_2. *Moreover, assume that there exists a* $\delta > 0$ *such that* $I \supset [t_0 - \delta, t_0 + \delta]$. *Then* (2.14) *has a unique solution* $u \in C^1([t_0 - T_0, t_0 + T_0], \mathbb{R}^d)$ *satisfying* $u|_{t=t_0} = u_0$, *where* $T_0 = \min(\delta, 1/(2C_2), C_1/C_3)$.

Proof. Similar to the proof of Theorem 2.1.5 we apply the contraction mapping Theorem 2.1.3 to the integrated ODE

$$u(t) = u_0 + \int_{t_0}^t f(u(s), s)\,\mathrm{d}s =: F(u)(t), \tag{2.15}$$

where $F : M \to M$ with

$$M = C^0([t_0 - T_0, t_0 + T_0], \{u \in \mathbb{R}^d \ : \ \|u - u_0\|_{\mathbb{R}^d} \leq C_1\})$$

which is equipped with the metric

$$d(u, v) = \sup_{t\in[t_0-T_0,t_0+T_0]} \|u(t) - v(t)\|_{\mathbb{R}^d} =: \|u - v\|_M.$$

We have

$$\begin{aligned}\|F(u) - u_0\|_M &\leq \sup_{t\in[t_0-T_0,t_0+T_0]} \|\int_{t_0}^t f(u(s), s)\,\mathrm{d}s\|_{\mathbb{R}^d} \\ &\leq \sup_{t\in[t_0-T_0,t_0+T_0]} |\int_{t_0}^t \|f(u(s), s)\|_{\mathbb{R}^d}\,\mathrm{d}s| \leq T_0C_3 \leq C_1\end{aligned}$$

for $T_0 = \min(\delta, C_1/C_3)$ such that F maps M into M. Moreover,

$$\begin{aligned}\|F(u) - F(v)\|_M &\leq \sup_{t\in[t_0-T_0,t_0+T_0]} \|\int_{t_0}^t f(u(s), s) - f(v(s), s)\,\mathrm{d}s\|_{\mathbb{R}^d} \\ &\leq \sup_{t\in[t_0-T_0,t_0+T_0]} |\int_{t_0}^t \|f(u(s), s) - f(v(s), s)\|_{\mathbb{R}^d}\,\mathrm{d}s| \\ &\leq \sup_{t\in[t_0-T_0,t_0+T_0]} |\int_{t_0}^t C_2\|u(s) - v(s)\|_{\mathbb{R}^d}\,\mathrm{d}s| \\ &\leq T_0C_2\|u - v\|_M,\end{aligned}$$

such that F is a contraction for $T_0 = \min(\delta, 1/(2C_2))$. From $u \in M$ it follows that $F(u) \in C^1((t_0 - T_0, t_0 + T_0), \mathbb{R}^d)$. Since u is a fixed point we also have $u = F(u) \in C^1((t_0 - T_0, t_0 + T_0), \mathbb{R}^d)$. □

Remark 2.2.2. The last argument shows that $f \in C^m(\mathbb{R}^d \times I, \mathbb{R}^d)$ implies $u \in C^{m+1}(I, \mathbb{R}^d)$. ⌋

For $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ locally Lipschitz-continuous the solutions can only stop to exist if $\|u(t)\|_{\mathbb{R}^d}$ becomes infinitely large.

Theorem 2.2.3. *For locally Lipschitz-continuous f the solution u with $u|_{t=t_0} = u_0 \in \mathbb{R}^d$ exists for all $t \in (T_-, T_+)$, where*

$$T_- = \inf\{t \in \mathbb{R} : \|u(t)\|_{\mathbb{R}^d} < \infty\} \quad \textit{and} \quad T_+ = \sup\{t \in \mathbb{R} : \|u(t)\|_{\mathbb{R}^d} < \infty\}.$$

Proof. If $\|u(t)\|_{\mathbb{R}^d}$ is finite, then the local existence and uniqueness Theorem 2.2.1 applies and also $\|u(t - T_0)\|_{\mathbb{R}^d}$ and $\|u(t + T_0)\|_{\mathbb{R}^d}$ are finite for some $T_0 > 0$. □

The following two examples show that f being only continuous is not sufficient for uniqueness, and that solutions in general do not exist globally.

Example 2.2.4. Consider the one-dimensional ODE $\dot{u} = \sqrt{|u|}$ with initial value $u|_{t=0} = 0$. The right-hand side is not Lipschitz-continuous at $u = 0$. This initial value problem has the solution $u = 0$, but also infinitely many other solutions, namely

$$u_\tau(t) = \begin{cases} 0, & \text{for} \quad 0 \le t \le \tau \\ (t-\tau)^2/4, & \text{for} \quad \tau \le t, \end{cases}$$

solves the ODE for each $\tau > 0$, i.e., there is no uniqueness of solutions. ⌋

Example 2.2.5. Consider the one-dimensional ODE $\dot{u} = 1 + u^2$ with the initial value $u|_{t=0} = 0$. This initial value problem has the solution $u(t) = \tan t$, i.e., the solution explodes, or blows up, in finite time, it becomes ∞ for $t = \pi/2$. Hence, the solution does not exist for all $t \in [0, \infty)$, i.e., there is no global existence of solutions. ⌋

Our next goal is to prove the continuity of the solutions w.r.t. the initial conditions. In order to do so we use Gronwall's inequality, cf. Lemma 2.1.7.

Lemma 2.2.6. *Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be continuous and locally Lipschitz-continuous w.r.t. the first variable. Then each solution $u(t, t_0, u_0)$ is Lipschitz-continuous w.r.t. u_0 in the following sense: For every $T_0 > 0$ there exist $\delta > 0$ and $L > 0$ such that for all $u_1 \in \mathbb{R}^d$ with $\|u_0 - u_1\|_{\mathbb{R}^d} \le \delta$ we have, for all $t \in [t_0 - T_0, t_0 + T_0]$,*

$$\|u(t, t_0, u_0) - u(t, t_0, u_1)\|_{\mathbb{R}^d} \le L \|u_0 - u_1\|_{\mathbb{R}^d}. \tag{2.16}$$

Proof. We have

$$\begin{aligned} &\|u(t, t_0, u_0) - u(t, t_0, u_1)\|_{\mathbb{R}^d} \\ &\qquad \le \|u_0 - u_1\|_{\mathbb{R}^d} + \int_{t_0}^t \|f(u(\tau, t_0, u_0), \tau) - f(u(\tau, t_0, u_1), \tau)\|_{\mathbb{R}^d} \, \mathrm{d}\tau \\ &\qquad \le \|u_0 - u_1\|_{\mathbb{R}^d} + |C_2 \int_{t_0}^t \|u(\tau, t_0, u_0) - u(\tau, t_0, u_1)\|_{\mathbb{R}^d} \, \mathrm{d}\tau|. \end{aligned}$$

Hence, by Gronwall's inequality we find

$$\|u(t, t_0, u_0) - u(t, t_0, u_1)\|_{\mathbb{R}^d} \le \|u_0 - u_1\|_{\mathbb{R}^d} \mathrm{e}^{C_2|t-t_0|}.$$

□

Remark 2.2.7. For continuously differentiable f we have differentiability of $u(t, t_0, u_0)$ w.r.t. the data t_0, u_0, see, e.g., [**HSD04**, Page 402]. ⌋

Another application of Gronwall's inequality is the proof of bounds on the size of the solutions. We state it in a differential form and then present a simple, but fundamental, example.

Lemma 2.2.8. (Gronwall) *Let $I \subset \mathbb{R}$ be an interval, $\alpha, \beta \in \mathbb{R}$, and $\phi \in C^1(I, \mathbb{R})$ a non-negative function with*

$$\dot{\phi}(t) \leq \alpha + \beta\phi(t)$$

for all $t \in I$. Then, for all $t_0, t \in I$, $t \geq t_0$ we have

$$\phi(t) \leq \phi(t_0)\mathrm{e}^{\beta(t-t_0)} + \frac{\alpha}{\beta}\left(\mathrm{e}^{\beta(t-t_0)} - 1\right).$$

Proof. We introduce $\psi(t) = \phi(t)\mathrm{e}^{\beta t}$ which satisfies $\dot{\psi}(t) \leq \alpha\mathrm{e}^{-\beta t}$. Integration yields $\psi(t) \leq \psi(t_0) + \frac{\alpha}{\beta}(\mathrm{e}^{-\beta t_0} - \mathrm{e}^{-\beta t})$. Undoing the transformation gives the result. □

Example 2.2.9. Consider $\dot{u} = u - u^3$. For $\phi(t) = u^2(t)$ we obtain

$$\dot{\phi}(t) = 2u(t)\dot{u}(t) = 2u^2(t) - 2u^4(t) \leq 2 - 2u^2(t) = 2 - 2\phi(t).$$

Therefore, from Lemma 2.2.8

$$u^2(t) \leq u^2(0)\mathrm{e}^{-2t} + 2(1 - \mathrm{e}^{-2t})/2 \searrow 1 \text{ as } t \to \infty.$$

Thus, every solution exists globally (in forward time) since it stays bounded and enters for instance the interval $[-2, 2]$. ⌋

As long as they exist, solutions of ODEs (2.14) have the trivial, but fundamental property

$$u(t+s, t_0, u_0) = u(t, s, u(s, t_0, u_0)), \quad u(t_0, t_0, u_0) = u_0. \tag{2.17}$$

For autonomous systems we have

$$u(t, t_0, u_0) = u(t - t_0, 0, u_0) =: u(t - t_0, u_0),$$

i.e., w.l.o.g. we can always choose the initial time $t_0 = 0$. Then (2.17) transfers into

$$u(t+s, u_0) = u(t, u(s, u_0)), \quad u(0, u_0) = u_0. \tag{2.18}$$

Thus, it makes no difference whether we solve the ODE until the time $t+s$, or if we solve the ODE until the time s, start again, and solve until the time $t+s$. A similar structure occurs for iterations $u_{n+1} = f(u_n)$. In the following we focus on the autonomous case.

Definition 2.2.10. *A map $u : I \times M \to M$ which satisfies* (2.18)*, where $I = \mathbb{R}_+$, $I = \mathbb{R}$, $I = \mathbb{N}$ or $I = \mathbb{Z}$, and where M is a set, is called a dynamical system or flow.*

If $I = \mathbb{R}_+$ or $I = \mathbb{R}$ then the dynamical system is called continuous.

If $I = \mathbb{N}$ or $I = \mathbb{Z}$ it is called discrete.

The set M is called the phase space.

The set $\gamma_+(u_0)=\{u(t,u_0) : t \geq 0\}$ is called the forward orbit through u_0, the set $\gamma_-(u_0) = \{u(t,u_0) : t \leq 0\}$ is called the backward orbit through u_0, and $\gamma(u_0) = \gamma_+(u_0) \cup \gamma_-(u_0)$ is called the orbit through u_0.

Since with this strict definition, only ODEs (2.14) with solutions existing globally forward in time in case $I = \mathbb{R}_+$, respectively, forward and backward in time in case $I = \mathbb{R}$ define continuous dynamical systems in the phase space $\mathbb{R}^d$, we shall not be that strict in the following and call any map u which fulfills (2.18) a dynamical system.

For ODEs (and PDEs) the dynamical systems property (2.18) expressed in terms of the family of (nonlinear) solution operators $(\mathcal{S}_t)_{t\in I}$ is given by

$$\mathcal{S}_{t+s} = \mathcal{S}_t\mathcal{S}_s, \quad \mathcal{S}_0 = \mathrm{I}, \tag{2.19}$$

where $\mathcal{S}_t$ is defined by $\mathcal{S}_t u_0 := u(t,u_0)$, and where I is here the identity on M. Due to (2.19), the family of solution operators $(\mathcal{S}_t)_{t\in I}$ is called a semigroup in case $I = \mathbb{R}^+$.

Remark 2.2.11. Except for the points which are mapped to infinity the map $u_0 \mapsto u(t,u_0)$ is bijective due to $u(t,u(-t,u_0))=u(t-t,u_0)=u_0$ in case $I = \mathbb{R}$ or $I = \mathbb{Z}$. Since additionally $u(t,u_0)$ depends continuously on u_0 dynamical systems can be interpreted as a flow of homeomorphisms, i.e., as flow of bijective bi-continuous maps from $\mathbb{R}^d$ into $\mathbb{R}^d$. If f is C^k then also $u(t,u_0)$ is C^k w.r.t. u_0, i.e., then the dynamical system can be interpreted as a flow of C^k-diffeomorphisms. ⌋

2.3. Special solutions

In this section we introduce special solutions such as fixed points, periodic solutions, and homoclinic and heteroclinic orbits and basic concepts such as stability and instability and invariant manifolds.

2.3.1. Fixed Points. Until further notice we consider the autonomous case. In order to explore the dynamics in phase space we start from the most simple dynamical objects, namely fixed points.

Definition 2.3.1. *A point $u^* \in \mathbb{R}^d$ is called fixed point for the ODE* (2.14) *if $f(u^*) = 0$.*

Example 2.3.2. We consider $\dot{u} = u - u^3$ with $u = u(t) \in \mathbb{R}$. From $f(u) = u - u^3 = 0$ we obtain the fixed points $u_1^* = 0$ and $u_{2,3}^* = \pm 1$. See Figure 2.4. ⌋

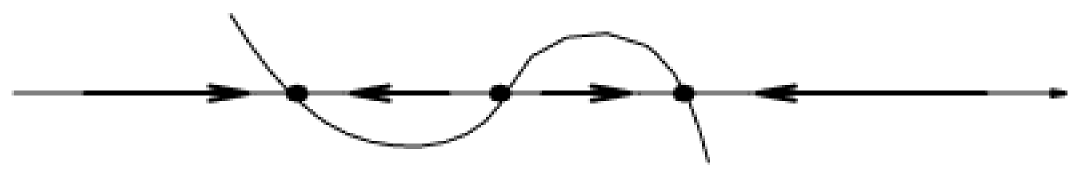


Figure 2.4. The phase portrait of $\dot{u} = u - u^3$ drawn in the 1D phase space. In one space dimension, i.e., $u = u(t) \in \mathbb{R}$, the complete qualitative behavior of the dynamics of an autonomous ODE is known with the knowledge of the fixed points due to topological reasons. In between two fixed points the "vector field" $f(u) \in \mathbb{R}$, which is a scalar function, cannot change sign. Therefore, the real line is divided by the fixed points which are connected by so called heteroclinic solutions. In case of an interval of fixed points the statement remains true with obvious modifications. Hence, the dynamics of autonomous one-dimensional ODEs is trivial.

In order to explore the dynamics near a fixed point u^* we write $u = u^* + v$ and make a Taylor expansion of the vector field f around u^*, i.e.,

$$\dot{v} = \frac{\mathrm{d}}{\mathrm{d}t}(u^* + v) = f(u^* + v) = 0 + \partial_u f(u^*)v + \mathcal{O}(\|v\|^2),$$

using $f(u^*) = 0$. The only approximate system which in general can be solved explicitly is the linearization at the fixed point u^*, namely

$$\dot{v} = \partial_u f(u^*)v.$$

In Example 2.3.2 we have $\partial_u f(u_j^*) = 1 - 3(u_j^*)^2$, and hence

$$\dot{v} = v \quad \text{for} \quad u_1^* = 0 \qquad \text{and} \qquad \dot{v} = -2v \quad \text{for} \quad u_{2,3}^* = \pm 1.$$

Therefore, from the linearization we expect that solutions which start close to $u_{2,3}^*$ converge towards $u_{2,3}^*$ for $t \to \infty$, while solutions which start close to u_1^* will leave any small neighborhood of u_1^*.

Definition 2.3.3. *A fixed point u^* is called stable for the ODE* (2.14) *if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|u_0 - u^*\|_{\mathbb{R}^d} < \delta$ implies $\|u(t, u_0) - u^*\|_{\mathbb{R}^d} < \varepsilon$ for all $t \geq 0$. Otherwise, it is called unstable. A stable fixed point is called asymptotically stable if $\|u_0 - u^*\|_{\mathbb{R}^d} < \delta$ additionally implies $\lim_{t\to\infty} u(t, u_0) = u^*$.*

For linear systems the statements of Theorem 2.1.14 remain true with the more general Definition 2.3.3. The following theorem guarantees in many situations that stability or instability in the linearized system implies stability or instability for the full system (2.14).

Theorem 2.3.4. *Let u^* be a fixed point for (2.14). Let $A = \partial_u f(u^*) \in \mathbb{R}^{d\times d}$ be the linearization of f in u^*.*

a) If all eigenvalues λ_j of A satisfy $\operatorname{Re}\lambda_j < 0$, then u^* is asymptotically stable.

b) If A has an eigenvalue λ with $\operatorname{Re}\lambda > 0$, then u^* is unstable.

Proof. W.l.o.g. let $u^* = 0$. The proof is based in both situations on the fact that the nonlinear terms in a neighborhood of $u^* = 0$ are much smaller than the linear terms.

a) In order to use this fact in a) we use the variation of constant formula. Let e^{tA} be the solution operator of the linear system $\dot{u} = Au$. Since the eigenvalues of A have strictly negative real part there are positive constants μ_0 and C_0 (necessary due to possible Jordan blocks) with

$$\|\mathrm{e}^{tA}\|_{\mathbb{R}^d\to\mathbb{R}^d} \le \|S\mathrm{e}^{tJ}S^{-1}\| \le \|S\|\|S^{-1}\|\|\mathrm{e}^{tJ}\| \le C_0\mathrm{e}^{-\mu_0 t}$$

for all $t \ge 0$ where we used the notation of (2.9). The estimate follows by using $\|S\|\|S^{-1}\| < \infty$ and by estimating $\|\mathrm{e}^{tJ}\|$ in the $\|\cdot\|^*$ matrix norm associated to the $\|\cdot\|_2$ vector norm from page 19. We remark that $-\mu_0$ has to be larger than the largest real part of the eigenvalues of A in case of Jordan blocks. The closer $-\mu_0$ gets to the largest real part, the larger the constant C_0 becomes.

For the nonlinear terms $g(u) = f(u) - Au = \mathcal{O}(\|u\|^2_{\mathbb{R}^d})$ the following holds: For all $b > 0$ there exists a $\delta_0 > 0$ such that $\|u\|_{\mathbb{R}^d} \le \delta_0$ implies $\|g(u)\|_{\mathbb{R}^d} \le b\|u\|_{\mathbb{R}^d}$.

The variation of constant formula, cf. (2.6),

$$u(t) = \mathrm{e}^{tA}u_0 + \int_0^t \mathrm{e}^{(t-s)A}g(u(s))\,\mathrm{d}s$$

then implies

$$\begin{aligned}\|u(t)\|_{\mathbb{R}^d} &\le \|\mathrm{e}^{tA}\|_{\mathbb{R}^d\to\mathbb{R}^d}\|u_0\|_{\mathbb{R}^d} + \int_0^t \|\mathrm{e}^{(t-s)A}\|_{\mathbb{R}^d\to\mathbb{R}^d}\ \|g(u(s))\|_{\mathbb{R}^d}\,\mathrm{d}s \\ &\le C_0\mathrm{e}^{-\mu_0 t}\|u_0\|_{\mathbb{R}^d} + \int_0^t C_0\mathrm{e}^{-\mu_0(t-s)}b\|u(s)\|_{\mathbb{R}^d}\,\mathrm{d}s,\end{aligned}$$

and as a consequence

$$\mathrm{e}^{\mu_0 t}\|u(t)\|_{\mathbb{R}^d} \le C_0\|u_0\|_{\mathbb{R}^d} + \int_0^t C_0\mathrm{e}^{\mu_0 s}b\|u(s)\|_{\mathbb{R}^d}\,\mathrm{d}s.$$

Gronwall's inequality, cf. Lemma 2.2.8, applied to $\mathrm{e}^{\mu_0 t}\|u(t)\|_{\mathbb{R}^d}$ finally implies $\mathrm{e}^{\mu_0 t}\|u(t)\|_{\mathbb{R}^d} \le C_0\|u_0\|_{\mathbb{R}^d}\mathrm{e}^{C_0 bt}$, respectively

$$\|u(t)\|_{\mathbb{R}^d} \le C_0\|u_0\|_{\mathbb{R}^d}\mathrm{e}^{(C_0 b-\mu_0)t}.$$

Choosing $b = \mu_0/(2C_0)$ defines $\delta_0 = \delta_0(b)$. Therefore, with $\mu = \mu_0/2$, we find $\|u(t)\|_{\mathbb{R}^d} \leq C_0\|u_0\|_{\mathbb{R}^d}\mathrm{e}^{-\mu t} \to 0$ for $t \to \infty$. Since this additionally implies that for any given $\varepsilon > 0$ we can choose $\delta = \frac{1}{2}\min\{C_0^{-1}\varepsilon, \delta_0\}$ such that $\|u_0\|_{\mathbb{R}^d} < \delta$ implies $\|u(t, u_0)\|_{\mathbb{R}^d} \leq C_0\delta < \varepsilon$, the asymptotic stability of $u^* = 0$ follows.

b) In order to prove b) we show that in the direction of the unstable subspaces there is a sector with radius ε which is entered by the solutions along the sides through the origin and which is left by the solutions along the side opposite to the origin. See Figure 2.5.

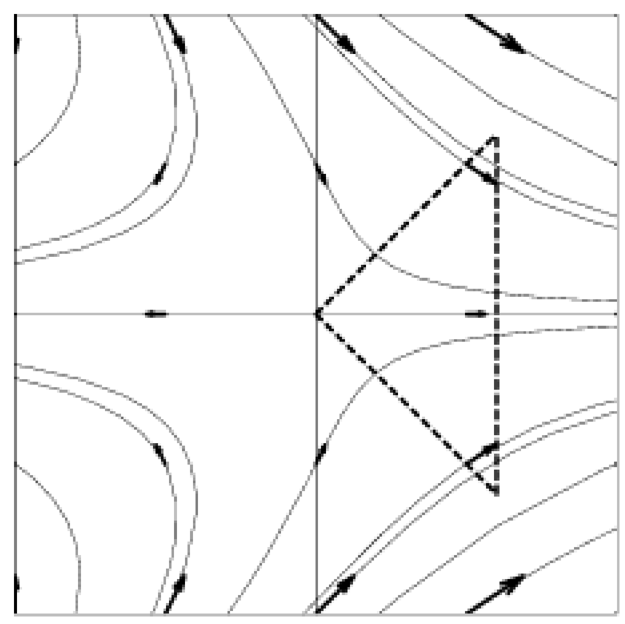


Figure 2.5. The phase portrait and the sector in the unstable case in a typical situation.

We start with a linear change of coordinates such that after the transform the linear part is of the form $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $A_1 \in \mathbb{R}^{k\times k}$ belongs to the part of the spectrum of A with positive real part and $A_2 \in \mathbb{R}^{(d-k)\times(d-k)}$ to the part of the spectrum of A with non-positive real part. Hence, there exists a $\sigma > 0$, such that for all eigenvalues λ_j of A_1 we have $\operatorname{Re}\lambda_j > \sigma$. Moreover, the change of coordinates is made in such a way that the norm of the off-diagonal elements of the transformed matrix A is less than γ. In order to do so, we assume $(\sum_{j=1}^k \sum_{m=1,m\neq j}^k |a_{jm}|^2)^{1/2} \leq \gamma$ for which we further assume $\gamma \leq \sigma/20$. From linear algebra it is known that this can always be achieved by using modified Jordan blocks. By changing the length of the vectors of the basis for instance the Jordan block $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ can be changed into $\begin{pmatrix} \lambda & r \\ 0 & \lambda \end{pmatrix}$ for every fixed $r > 0$. Like in a) we find for the nonlinear terms that for all $b > 0$ there exists a $\delta_0 > 0$ such that $\|u\|_{\mathbb{R}^d} \leq \delta_0$ implies $\|g(u)\|_{\mathbb{R}^d} \leq b\|u\|_{\mathbb{R}^d}$.

Next we define $R^2 = \sum_{j=1}^k |u_j|^2$ and $\rho^2 = \sum_{j=k+1}^d |u_j|^2$ and assume that $u^* = 0$ is stable. Then for all $\varepsilon > 0$ there exists a $\delta > 0$, such that $\rho(0) + R(0) < \delta$ implies $\rho(t) + R(t) < \varepsilon$ for all $t \geq 0$. For the transformed system we find

$$2R\frac{\mathrm{d}}{\mathrm{d}t}R = \frac{\mathrm{d}}{\mathrm{d}t}(R^2) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{k}|u_j|^2 = 2\mathrm{Re}\sum_{j=1}^{k}\overline{u}_j\partial_t u_j$$
$$= 2\mathrm{Re}\sum_{j=1}^{k}\overline{u}_j(\lambda_j u_j + \sum_{m\neq j} a_{jm}u_m + g_j).$$

Using

$$2\mathrm{Re}\sum_{j=1}^{k}\overline{u}_j\lambda_j u_j \geq 2\sigma\sum_{j=1}^{k}\overline{u}_j u_j = 2\sigma R^2$$

and

$$|2\mathrm{Re}\sum_{j=1}^{k}\overline{u}_j\sum_{m\neq j}a_{jm}u_m| \leq 2(\sum_{j=1}^{k}\sum_{m=1,m\neq j}^{k}|a_{jm}|^2)^{1/2}R^2 \leq 2\gamma R^2,$$

together with

$$|2\mathrm{Re}\sum_{j=1}^{k}\overline{u}_j g_j| \leq 2Rb\sqrt{\rho^2+R^2} \leq 2Rb(\rho+R)$$

yields

$$2R\frac{\mathrm{d}}{\mathrm{d}t}R \geq 2\sigma R^2 - 2\gamma R^2 - 2bR(\rho+R).$$

Choosing $b = \sigma/10$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}R \geq \sigma R/2 - b\rho.$$

Similarly, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho \leq \sigma\rho/20 + b(\rho+R).$$

where we used $2\mathrm{Re}\sum_{j=k+1}^{d}\overline{u}_j\lambda_j u_j \leq 0$. Since

$$\sigma R/2 - b\rho - \sigma\rho/20 - b(\rho+R) \geq \sigma(R-\rho)/4$$

we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(R-\rho) \geq \sigma(R-\rho)/4$$

and as consequence

$$R(t) - \rho(t) \geq (R(0)-\rho(0))\mathrm{e}^{\sigma t/4}.$$

For solutions with $R(0) = 2\rho(0)$ it follows that $R(t) \geq \rho(0)\mathrm{e}^{\sigma t/4}$. However, this contradicts the assumption of stability, since $R(t)+\rho(t) \leq \varepsilon$ for all $t \geq 0$ is not possible, independent of how small $\rho(0) > 0$ or $\delta > 0$ has been chosen. □

Example 2.3.5. We consider again the situation from Example 2.3.2. As a consequence of Theorem 2.3.4, the linear stability analysis is sufficient to determine the stability of the fixed points in the nonlinear system. Hence, the fixed point $u_1^* = 0$ is unstable, since the linearization $A = 1 \in \mathbb{R}^{1\times 1}$ has the eigenvalue 1, whereas the fixed points $u_{2,3}^* = \pm 1$ are asymptotically stable, since $A = -2 \in \mathbb{R}^{1\times 1}$ has the eigenvalue -2. ⌋

Example 2.3.6. An example of an ODE with a fixed point in the origin which is stable in the linearized system, but unstable in the full system is given by $\dot{u} = u^3$. ⌋

The following theorem states that near a hyperbolic fixed point the flow can be completely linearized by a change of coordinates h. For a proof we refer to [**Tes12**, Theorem 9.9].

Definition 2.3.7. *A fixed point u^* is called hyperbolic, if the linearization $A = \partial_u f(u^*)$ has no eigenvalues with* $\mathrm{Re}\,\lambda = 0$.

Theorem 2.3.8. (Hartman-Grobman) *Let u^* be a fixed point for* (2.14), *let $\mathcal{S}_t$ be the flow of* (2.14), *let $A = \partial_u f(u^*) \in \mathbb{R}^{d\times d}$ be the linearization of f in u^*, and assume that A has no eigenvalues with zero real part. Then there exists an homeomorphism h from a neighborhood U of u^* to a neighborhood V of u^* such that for all $u_0 \in U$ there exists an open interval $I_0 \subset \mathbb{R}$, $0 \in I_0$ such that for all $t \in I_0$ we have*

$$h \circ \mathcal{S}_t u_0 = \mathrm{e}^{tA} h(u_0).$$

Thus, h maps trajectories of (2.14) *near u^* to trajectories of the linearization $\dot{y} = Ay$.*

Remark 2.3.9. There is also a discrete version of the Hartman-Grobman theorem: Consider the nonlinear map $u_{n+1} = f(u_n)$ and assume that $f(u^*) = u^*$ and that all eigenvalues λ of linearization $A = \partial_u f(u^*)$ around u^* satisfy $|\lambda| \neq 1$. Then there exists a homeomorphism h in a neighborhood U of u^* such that $h(f(u)) = Ah(u)$ for all $u \in U$. ⌋

It is somewhat surprising that even for analytic f the map h is in general not differentiable; see Exercise 2.9.

2.3.2. Periodic solutions. The first non-trivial dynamical object is a periodic solution.

Definition 2.3.10. *A solution $u = u(t)$ of the ODE* (2.14) *is called periodic if $u(t+T) = u(t)$ for a $T > 0$ and all $t \in \mathbb{R}$. If moreover, $u(t) \neq u(t+\tau)$ for all $0 < \tau < T$, then T is called the minimal period.*

Example 2.3.11. We consider the two-dimensional ODE

$$\begin{aligned}\dot{u}_1 =& -u_2 + u_1(1 - u_1^2 - u_2^2),\\ \dot{u}_2 =& u_1 + u_2(1 - u_1^2 - u_2^2).\end{aligned}$$

By introducing polar coordinates $u_1 = r\cos(\phi)$, $u_2 = r\sin(\phi)$ we obtain

$$\dot{r} = r - r^3 \quad \text{and} \quad \dot{\phi} = 1.$$

In order to understand the dynamics of this ODE we visualize its flow in the phase plane.

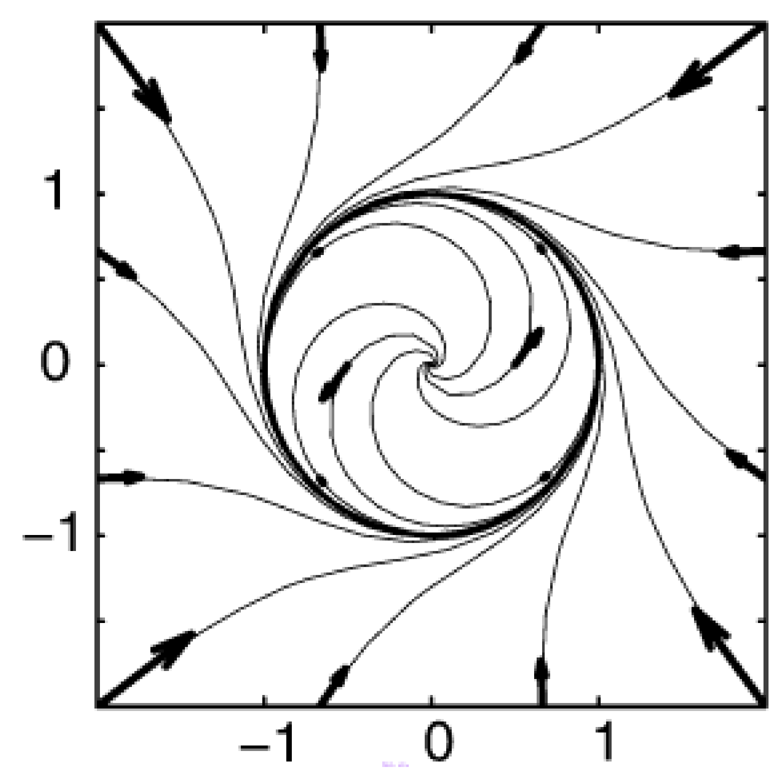


Figure 2.6. Flow for $\dot{u}_1 = -u_2 + u_1(1 - u_1^2 - u_2^2)$ and $\dot{u}_2 = u_1 + u_2(1 - u_1^2 - u_2^2)$.

From the phase portrait we find that all solutions converge towards the circle $r = 1$ which is a periodic solution with the minimal period $T = 2\pi$. Moreover, the origin $r = 0$ is an unstable fixed point in the r equation. As an exercise, we may consider the linearization around $(u_1, u_2) = 0$. We obtain

$$\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix} = A\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},$$

with eigenvalues $\lambda_{1,2} = 1 \pm \mathrm{i}$. Since $\operatorname{Re}\lambda_{1,2} = 1 > 0$ we also have with Theorem 2.3.4 the instability of the origin. On the other hand, from the phase portrait the periodic solution $r = 1$ seems to be asymptotically stable. However, as we see in a moment we have to be more precise when we talk about stability of periodic solutions. ⌋

The stability or instability of non-trivial periodic solutions is a non-trivial task due to the fact that the derivative $\dot{u}_{per}$ of the periodic orbit u_{per} solves the linearization $\dot{v} = Df(u_{per})v$ around the periodic orbit u_{per}. Hence, the linearization possesses a Floquet exponent with real part zero, and so even a generalization of Theorem 2.3.4 to non-autonomous systems would not be applicable for proving stability in the nonlinear system. In order to study stability of periodic solutions we proceed as follows. We introduce a so called Poincaré section, a hyperplane which intersects the periodic orbit transversally. For our example we choose for instance

$$S = \{(x, y) \ : \ u_2 = 0, \ u_1 \in (1/2, 3/2)\}.$$

Transversality means that in the intersection point $u^* = (u_1, u_2) = (1, 0)$ the Poincaré section S and the vector field $f(1, 0) = (0, 1)$ span the complete phase space $\mathbb{R}^2$. Then we define the so called Poincaré map $\Pi : S \to S$ as follows: for $u_0 \in S$ we let $\Pi(u_0)$ be the first intersection point of $t \mapsto u(t, u_0)$ and S for $t > 0$, i.e., in the example $\Pi(u_0) = u(2\pi, u_0)$. As Figure 2.7 illustrates, Poincaré maps to different Poincaré sections are conjugated to each other in the following sense. Let Π_{S_1,S_2} be the map from section S_1 to S_2. Then we have $\Pi_{S_1,S_1} = \Pi_{S_1,S_2} \circ \Pi_{S_2,S_2} \circ \Pi_{S_2,S_1}$.

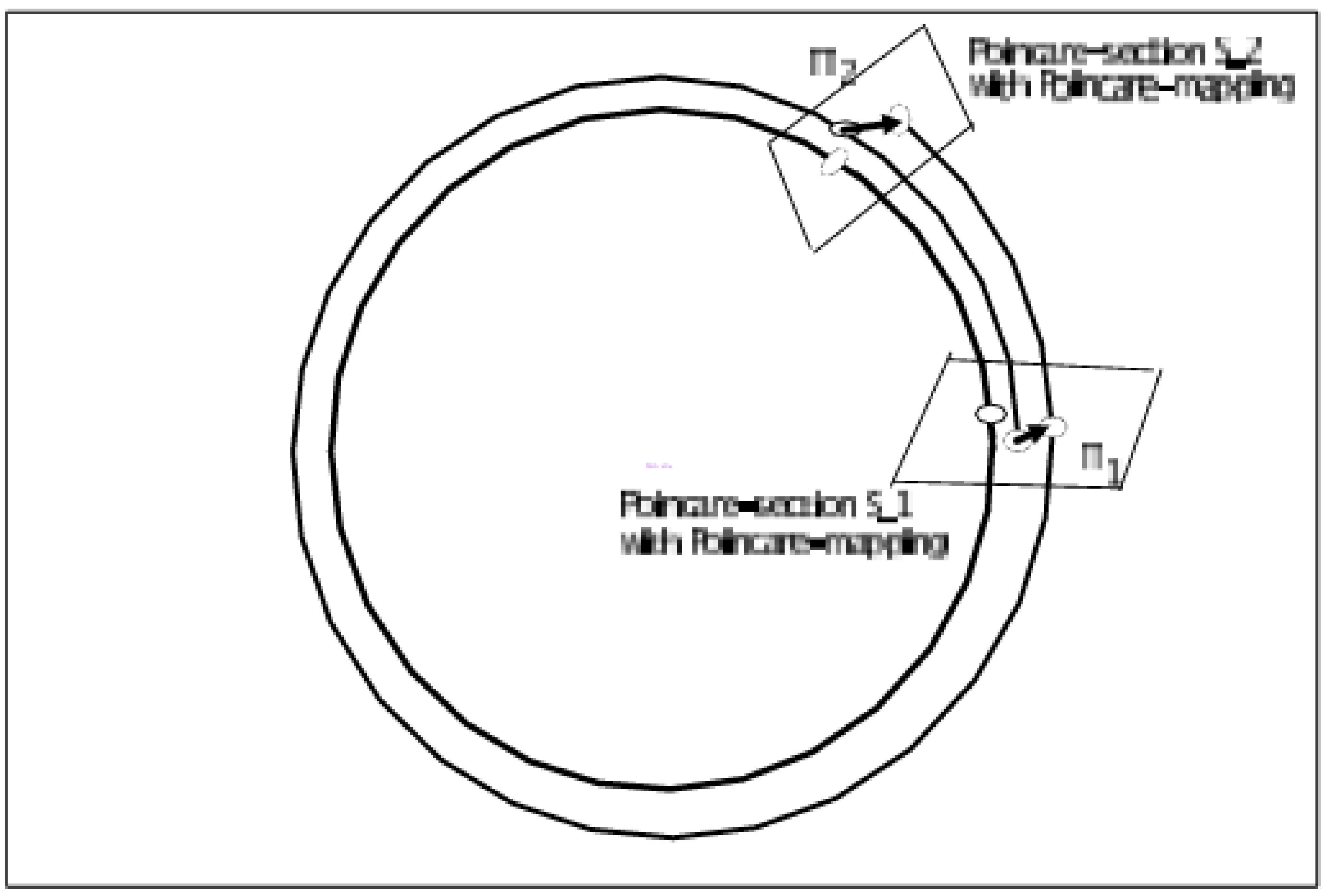


Figure 2.7. Two Poincaré maps to different Poincaré sections are conjugated to each other.

This fact and the fact that the intersection point u^* of the periodic solution is a fixed point of the Poincaré map Π, i.e., $\Pi(u^*) = u^*$, lead to the following definition.

Definition 2.3.12. *a) A fixed point u^* is called stable for the iteration $u_{n+1} = \Pi(u_n)$ with $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|u_0 - u^*\|_{\mathbb{R}^d} < \delta$ implies $\|\Pi^n(u_0) - u^*\|_{\mathbb{R}^d} < \varepsilon$ for all $n \in \mathbb{N}$. Otherwise, it is called unstable. A stable fixed point is called asymptotically stable if additionally $\lim_{n\to\infty} \Pi^n(u_0) = u^*$ holds.*
b) A periodic solution for the ODE (2.14) *is called stable, unstable, or asymptotically stable if the fixed point of the associated Poincaré map is stable, unstable, or asymptotically stable.*

The eigenvalues of the linearization $D\Pi$ of the Poincaré map now play an analogous role as the eigenvalues of the linearization A around a fixed point, cf. Theorem 2.3.4.

Definition 2.3.13. *The eigenvalues of the linearization $D\Pi$ of the Poincaré map are called Floquet multipliers.*

Theorem 2.3.14. *Let u^* be a fixed point of the map $\Pi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ and let $A = D\Pi \in \mathbb{R}^{d-1\times d-1}$ be the linearization of Π in u^*.*

a) If all eigenvalues λ_j of $D\Pi$ satisfy $|\lambda_j| < 1$, then u^* is asymptotically stable.

b) If A has an eigenvalue λ with $|\lambda| > 1$, then u^* is unstable.

Proof. The proof goes along the lines of the proof of Theorem 2.3.4. □

Example 2.3.15. In order to prove the stability of the periodic solution $r = 1$ in Example 2.3.11 it remains to compute the Floquet multipliers. Since $\widetilde{r} = r - 1$ satisfies $\dot{\widetilde{r}} = -2\widetilde{r} + \mathcal{O}(\widetilde{r}^2)$ and since $\Pi(u_0) = u(2\pi, u_0)$ for $u_0 \in S$ due to ϕ solving $\dot{\phi} = 1$, we find

$$D\Pi((1,0)) = \mathrm{e}^{-2\cdot 2\pi} \in \mathbb{R}^{1\times 1}.$$

Thus we have one Floquet multiplier with $|\mathrm{e}^{-4\pi}| < 1$, which implies the stability of the periodic solution $r = 1$. ⌋

2.3.3. Homoclinic and heteroclinic solutions. Homoclinic and heteroclinic solutions connect fixed points with themselves or other fixed points. Pulse and front solutions in PDEs correspond to homoclinic and heteroclinic solutions in associated ODEs.

Definition 2.3.16. *A solution $u = u(t)$ of the ODE (2.14) is called heteroclinic, if $u_+ \neq u_-$, or homoclinic, if $u_+ = u_-$, connection between the fixed points u_- and u_+ if $\lim_{t\to-\infty} u(t) = u_-$ and $\lim_{t\to\infty} u(t) = u_+$.*

Homoclinic and heteroclinic solutions converge to the fixed points along special sets, namely to the fixed point u_- along the unstable manifold of u_- for $t \to -\infty$, and to the fixed point u_+ along the stable manifold of u_+ for $t \to \infty$.

Definition 2.3.17. *Let u^* be a fixed point of the ODE (2.14). The set*

$$W_s = \{u_s \in \mathbb{R}^d : \exists\ \beta > 0\ :\ \lim_{t\to\infty} \|u(t, u_s) - u^*\|_{\mathbb{R}^d} \mathrm{e}^{\beta t} = 0\}$$

is called the stable manifold of u^*. The set

$$W_u = \{u_u \in \mathbb{R}^d : \exists\ \beta > 0\ :\ \lim_{t\to-\infty} \|u(t, u_u) - u^*\|_{\mathbb{R}^d} \mathrm{e}^{\beta |t|} = 0\}$$

is called the unstable manifold of u^*.

Example 2.3.18. For $\dot{u}_1 = -u_1,\ \dot{u}_2 = u_2$ we have $W_s = \{(u_1, 0) : u_1 \in \mathbb{R}\}$ and $W_u = \{(0, u_2) : u_2 \in \mathbb{R}\}$. ⌋

The following theorem guarantees that the sets W_s and W_u from Definition 2.3.17 are smooth manifolds and that they are invariant under the flow of the ODE.

Theorem 2.3.19. (Invariant manifolds) *Let u^* be a fixed point of the ODE (2.14), $f \in C^k(\mathbb{R}^d, \mathbb{R}^d)$, and let $A = \partial_u f(u^*) \in \mathbb{R}^{d\times d}$ be the linearization of f in u^*. Let*

$$E_s = \operatorname{span}\{\varphi : \varphi \ \textit{ eigenvector of } A \textit{ to eigenvalues } \lambda \textit{ with } \operatorname{Re}\lambda < 0\}$$

be the so called stable subspace and let

$$E_u = \operatorname{span}\{\varphi : \varphi \ \textit{ eigenvector of } A \textit{ to eigenvalues } \lambda \textit{ with } \operatorname{Re}\lambda > 0\}$$

be the so called unstable subspace, where eigenvectors include here in all cases generalized eigenvectors. Then there exists a unique C^k-manifold $W_s = W_s(u^*)$ tangential to the stable subspace E_s, which coincides with the stable manifold from Definition 2.3.17, and a unique C^k-manifold $W_u = W_u(u^*)$ tangential to the unstable subspace E_u, which coincides with the stable manifold from Definition 2.3.17. Moreover, there exists a (non-unique) C^{k-1}-center manifold tangential to the center subspace

$$E_c = \operatorname{span}\{\varphi : \varphi \textit{ eigenvector of } A \textit{ to eigenvalues } \lambda \textit{ with } \operatorname{Re}\lambda = 0\}.$$

If $f \in C^\infty$, then $W_s, W_u \in C^\infty$. The center manifold W_c can be chosen to be in C^r for all $r < \infty$. All these manifolds are invariant under the flow of the ODE (2.14). A set M is called invariant if $u_0 \in M$ implies $u(t, u_0) \in M$ for all $t \in \mathbb{R}$.

Proof. The lengthy proof of this theorem is well documented in [**Van89**]. A sketch of the proof of the existence of the center manifold can be found in §13.1. □

Example 2.3.20. Consider the equations for the mathematical pendulum without friction, namely

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= -\sin(u_1), \end{aligned} \tag{2.20}$$

where u_1 is the angle between the pendulum and the vertical axis. We find the fixed points $u^* = (k\pi, 0)$ for $k \in \mathbb{Z}$. The linearization at $(k\pi, 0)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos(k\pi) & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

which yields the eigenvalues $\lambda \in \begin{cases} \{-1,1\} & \text{for } k \in 2\mathbb{Z}+1 \\ \{-\mathrm{i},\mathrm{i}\} & \text{for } k \in 2\mathbb{Z} \end{cases}$ implying saddles for odd k and centers for even k. The eigenvectors at the saddles are $\varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, in the stable direction, and unstable direction, respectively. Physical intuition lets us suspect the existence of, for instance, heteroclinic orbits

$$\gamma_+ = W_s((\pi,0)) \cap W_u((-\pi,0)) \quad \text{and} \quad \gamma_- = W_u((\pi,0)) \cap W_s((-\pi,0)),$$

corresponding to one complete rotation of the pendulum from the unstable upper rest state to itself in infinite time. Since the points $(k\pi,0)$ and $((k+2)\pi,0)$ can be identified, γ_-, γ_+ can also be called homoclinic. These two orbits separate $\mathbb{R}^2$ into two domains, a bounded domain inside and an unbounded domain outside the two orbits. We further suspect the domain inside to be filled with periodic orbits corresponding to oscillations of the pendulum with amplitude $< \pi$. See Figure 2.8.

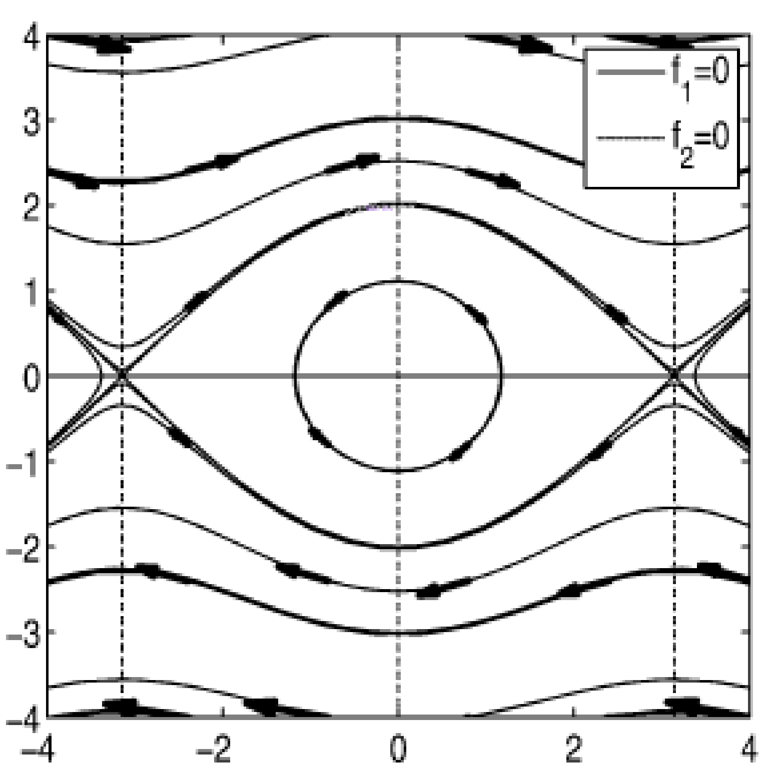


Figure 2.8. Phase portrait for the undamped pendulum.

The whole interior of the "eye" is filled with periodic solutions. Therefore, each of them is stable in the sense of Definition 2.3.12 with Floquet multiplier 1. ⌋

Remark 2.3.21. Instead of "physical intuition" we should rather use the fact that (2.20) is a Hamiltonian system, see also Chapter 4. Here, (2.20) is the first order system belonging to the second order equation

$$\ddot{u} = -\sin(u) = f(u) \tag{2.21}$$

corresponding to Newton's law, namely that the change of momentum equals the acting force. The qualitative behavior of equations such as (2.20) can be obtained by the following procedure independently of the concrete form of f. Multiplying (2.21) by $\dot{u}$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{u}^2 - F(u)\right) = 0, \quad \text{where} \quad F' = f. \tag{2.22}$$

Here $\frac{1}{2}(\dot{u})^2$ and $-F(u)$ are called the kinetic and potential energy, respectively, and (2.22) shows that the total energy $E = \frac{1}{2}(\dot{u})^2 - F(u)$ is conserved. Hence, orbits of (2.20) lie on level sets of E. It turns out that all equations from classical mechanics without friction can be written as Hamiltonian systems in the form

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = J\nabla H(q,p), \quad \text{where} \quad J = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \tag{2.23}$$

is skew symmetric, where $q \in \mathbb{R}^d$ and $p \in \mathbb{R}^d$ are the position and the momentum coordinates, and where $H : \mathbb{R}^{2d} \to \mathbb{R}$ is called the Hamiltonian, see Chapter 4. Also PDEs can have a Hamiltonian structure, see for instance §8.1 and §8.2. ⌋

Stable, center, and unstable manifolds can also be generalized from fixed points to more complicated objects, for instance to periodic solutions. They exist for discrete dynamical systems, too.

Remark 2.3.22. Let u^* be a fixed point of the iteration $u_{n+1} = \Pi u_n$. The set

$$W_s = \{u_s \in \mathbb{R}^d : \exists\, \beta > 0 \ : \ \lim_{n\to\infty} \|\Pi^n(u_s) - u^*\|_{\mathbb{R}^d} \mathrm{e}^{\beta n} = 0\}$$

is called the stable manifold of u^*. The set

$$W_u = \{u_u \in \mathbb{R}^d : \exists\, \beta > 0 \ : \ \lim_{n\to-\infty} \|\Pi^n(u_u) - u^*\|_{\mathbb{R}^d} \mathrm{e}^{\beta |n|} = 0\}$$

is called the unstable manifold of u^*. They exist as smooth invariant manifolds with similar properties as the ones explained in Theorem 2.3.19. ⌋

2.4. ω-limit sets and attractors

In this section we are interested in objects which describe the dynamics for $t \to \infty$. These are so called ω-limit sets and attractors. We characterize them for two-dimensional autonomous systems and gradient systems. The concepts of this section will later on be applied in Part II of this book to PDEs on spatially bounded domains, which very often can be written as countably infinite-dimensional dynamical systems. Therefore, throughout this subsection, we consider a general dynamical system $X \ni u_0 \mapsto \mathcal{S}_t u_0$, where X is some possibly infinite-dimensional Banach space. The theory has to be modified in Part III and Part IV where PDEs on unbounded domains will be handled. For simplicity, the reader may think of $\mathcal{S}_t$ as being defined by the solutions of some ODE $\dot{u} = f(u)$ in the phase space $X = \mathbb{R}^d$. This section follows rather closely [**Rob01**, §10], including a number of examples.

2.4.1. ω-limit sets. Given some initial condition u_0 for a dynamical system $\mathcal{S}_t u_0 = u(t, u_0)$ on X, the behavior of the solution for $t \to \infty$ is described by the ω-limit set, defined by

$$\omega(u_0) = \{v \in X : \exists\ (t_n)_{n\in\mathbb{N}} \text{ with } t_n \to \infty \text{ and } \lim_{n\to\infty} u(t_n, u_0) = v\}. \tag{2.24}$$

Thus, the ω-limit set of u_0 consists of all limit points of the forward orbit through u_0. Hence, an equivalent characterization is

$$\omega(u_0) = \cap_{t\geq 0}\overline{\cup_{s\geq t} u(s, u_0)}. \tag{2.25}$$

If $\gamma_+ = \gamma_+(u_0)$ is the (forward) orbit through u_0 and $v \in \gamma_+$ then $\omega(v) = \omega(u_0)$ such that we also write $\omega(\gamma_+) := \omega(u_0)$.

Theorem 2.4.1. *The set $\omega(\gamma_+)$ is closed and invariant. If $X = \mathbb{R}^d$ and γ_+ is bounded, then $\omega(\gamma_+)$ is compact, connected and non-empty. In general, if $\overline{\cup_{t\geq t_0}\mathcal{S}_t u_0}$ is compact for some $t_0 \geq 0$, then $\omega(\gamma_+)$ is compact, connected and non-empty.*

Proof. We first consider the case $X = \mathbb{R}^d$.

a) $\omega(\gamma_+)$ as set of limit points is closed.

b) Next, we prove the invariance. Let $p \in \omega(\gamma_+)$. Then there exists a sequence $t_n \to \infty$ such that $\lim_{t_n\to\infty} u(t_n) = p$. We have to prove that $u(t, p) \in \omega(\gamma_+)$. Since $u(t + t_n, x_0) = u(t, u(t_n, x_0))$ it follows for $n \to \infty$ that

$$u(t + t_n, x_0) \to u(t, p),$$

which implies that $\gamma(p) \subset \omega(\gamma_+)$.

c) With γ_+ bounded, $\omega(\gamma_+)$ is bounded. Since $\omega(\gamma_+)$ is closed by a), compactness follows.

d) Suppose that γ_+ consists of more than one point, i.e., γ_+ is not a fixed point. Then γ_+ consists of infinitely many points. Hence, there exists at least one limit point p of the bounded set γ_+. Suppose that γ_+ consists only of a fixed point. Then $\gamma_+ = \omega(\gamma_+)$ is also non-empty.

e) Suppose that $\omega(\gamma_+)$ is not connected, i.e., there exist closed sets A_1 and A_2 satisfying $\omega(\gamma_+) = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Since γ_+ is bounded, there exists a $R > 0$, such that $\gamma_+ \subset B_R(0) = \{x \in \mathbb{R}^d : \|x\| \leq R\}$. Let $\delta > 0$ be the distance between A_1 and A_2. Define

$$A_3 = \{u \in B_R(0) : \delta/4 \leq \operatorname{dist}(u, \omega(\gamma_+))\}$$

where $\operatorname{dist}(u, A) = \inf_{a\in A} \|u - a\|$. Obviously the solution must pass A_3 infinitely many times, and hence there must be a limit point in A_3 which contradicts the assumption $\omega(\gamma) = A_1 \cup A_2$.

In the general case, i.e., with X some Banach space, the proof works the same way, with $\omega(\gamma_+)$ compact as a closed subset of the compact set $\overline{\cup_{t\geq t_0}\mathcal{S}_t u_0}$. Note that this compactness argument is used in d) and e). $\square$

2.4.2. Attractors. Attractors are compact sets describing the asymptotic dynamics of the system in the limit $t \to \infty$. They exist for so called dissipative systems.

Definition 2.4.2. *The flow $\mathcal{S}_t : X \to X$ is called dissipative if there exists a compact set B such that for any bounded set $M \subset X$ there exists a $t_0 = t_0(M)$ such that $\mathcal{S}_t M \subset B$ for all $t \geq t_0$. The set B is then called absorbing.*

The goal is to define the so called global attractor $\mathcal{A}$ which contains as much information as possible about the asymptotic behavior for $t \to \infty$. If the system is dissipative and thus has a compact absorbing set B, a first idea would be to take $\cup_{u\in B}\omega(u)$. However, this set in general does not contain homo- or heteroclinic connections, which we already know to be relevant for the asymptotic dynamics. Therefore, a better choice turns out to be

$$\begin{aligned} \mathcal{A} = \omega(B) \;&=\; \{v \in X : \exists\, t_n \to \infty,\ \exists\, (u_n)_{n\in\mathbb{N}} \subset B, \\ &\qquad \text{with } \lim_{n\to\infty} u(t_n, u_n) = v\}. \end{aligned} \tag{2.26}$$

Definition 2.4.3. *A set $\mathcal{A} \subset X$ is called an attractor of the dynamical system $\mathcal{S}_t : X \to X$ if*

(i) $\mathcal{A}$ is compact and invariant;

(ii) there is a neighborhood U of $\mathcal{A}$ such that $\mathcal{A}$ attracts U.

The basin of attraction of $\mathcal{A}$ is defined as

$$B(\mathcal{A}) = \{u \in X : \operatorname{dist}(\mathcal{S}_t u, \mathcal{A}) \to 0 \text{ as } t \to \infty\},$$

where $\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|$. A set $\mathcal{A} \subset X$ is called the global attractor for $\mathcal{S}_t$ if additionally

(iii) $\mathcal{A}$ attracts all points in X, i.e., we have $B(\mathcal{A}) = X$.

Theorem 2.4.4. *If the flow $\mathcal{S}_t$ is dissipative and if B is a compact absorbing set, then $\mathcal{A} = \omega(B)$ is the unique global attractor. Moreover, $\mathcal{A}$ is connected, the maximal compact invariant set, and the minimal set that attracts all bounded sets.*

Proof. The fact that $\omega(B)$ is nonempty, compact, invariant and connected follows as in Theorem 2.4.1. To show that $\mathcal{A}$ is the maximal compact invariant set, let Y be a compact and invariant set. Then $\mathcal{S}_t Y = Y$ and, since B is absorbing, $\mathcal{S}_t Y \subset B$ for $t \geq t_0$, hence $Y \subset B$ and therefore $\omega(Y) = Y \subset \mathcal{A} = \omega(B)$. This shows that $\mathcal{A}$ is the maximal compact invariant set.

Next we show that $\mathcal{A}$ attracts all bounded sets. Suppose that this is not the case, then there is a bounded set Y, a $\delta > 0$, and a sequence $t_n \to \infty$ with

$$\operatorname{dist}(\mathcal{S}_{t_n} Y, \mathcal{A}) \geq \delta.$$

Thus, there are $u_n \in Y$ with $\operatorname{dist}(\mathcal{S}_{t_n} u_n, \mathcal{A}) \geq \delta/2$. Since Y is bounded and B absorbing we have $\mathcal{S}_{t_n} u_n \in B$ for n large enough, and as B is compact there is a subsequence with

$$\mathcal{S}_{t_{n_j}} u_{n_j} \to v \in B \quad \text{and} \quad \operatorname{dist}(v, \mathcal{A}) \geq \delta/2. \tag{2.27}$$

However, with $v_j = \mathcal{S}_{t_0} u_{n_j} \in B$ we have

$$v = \lim_{j\to\infty} \mathcal{S}_{t_{n_j}} u_{n_j} = \lim_{j\to\infty} \mathcal{S}_{t_{n_j}-t_0} S_{t_0} u_{n_j}$$

and hence $v \in \mathcal{A}$, which contradicts (2.27). Obviously, $\mathcal{A}$ is also the minimal set that attracts all bounded sets since $\mathcal{S}_t\mathcal{A} = \mathcal{A}$. □

Example 2.4.5. Consider $\dot{u} = u - u^3$. The attractor is given by $\mathcal{A} = [-1, 1]$ and contains the heteroclinic connections between 0 and ± 1. Every set $[-1-\delta, 1+\delta]$ with $\delta > 0$ is an absorbing set. ⌋

2.4.3. Shadowing and upper-semicontinuity of attractors. Clearly, an important issue is the relation of the flow $\mathcal{S}_t$ in X to that on the attractor. The following theorem roughly says that given an initial condition in X, there exists a $\tau > 0$ such that after the time τ the flow can be approximated by a flow on the attractor for some finite time.

Theorem 2.4.6. *Let $\mathcal{A}$ be the global attractor for the flow $\mathcal{S}_t$ and let $u_0 \in X$. For all $\varepsilon > 0$ and $T > 0$ there exists a $\tau = \tau(\varepsilon, T) > 0$ and a point $v_0 \in \mathcal{A}$ such that*

$$\|\mathcal{S}_{\tau+t} u_0 - \mathcal{S}_t v_0\|_X \leq \varepsilon \quad \textit{for all} \quad 0 \leq t \leq T.$$

Proof. From the continuous dependence on the initial conditions for given $\varepsilon, T > 0$ there exists a $\delta = \delta(\varepsilon, T)$ such that

$$\|u_1 - v_0\| \leq \delta \quad \Rightarrow \quad \|\mathcal{S}_t u_1 - \mathcal{S}_t v_0\| \leq \varepsilon \quad \text{for } t \in [0, T].$$

Since $\mathcal{A}$ is the global attractor, for any $u_0 \in X$ and every $\delta > 0$ there exists a time τ and a $v_0 \in \mathcal{A}$ such that $\|u_1 - v_0\| \leq \delta$ where $u_1 = S(\tau)u_0$. □

An approximation of a solution $(\mathcal{S}_t u_0)_{t\geq\tau}$ by a single solution in the attractor cannot be expected in general. However, solutions can be approximated (or shadowed) by so called pseudo-orbits in the attractor. Moreover, due to the attractivity property of $\mathcal{A}$ the approximation becomes better and better for larger and larger times.

Corollary 2.4.7. (Shadowing) *For all $u_0 \in X$ there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of errors $\varepsilon_n > 0$ with $\varepsilon_n \to 0$, an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of times with $t_{n+1} - t_n \to \infty$ for $n \to \infty$, and a sequence $(v_n)_{n\in\mathbb{N}}$ of points $v_n \in \mathcal{A}$ such that*

$$\|\mathcal{S}_t u_0 - \mathcal{S}_{t-t_n} v_n\|_X \le \varepsilon_n \quad \text{for all } t_n \le t \le t_{n+1},$$

and $\|v_{n+1} - \mathcal{S}_{t_{n+1}-t_n} v_n\|_X \to 0$ for $n \to \infty$.

However, in general a flow $\mathcal{S}_t u_0$ cannot be approximated by a single flow on the attractor as $t \to \infty$.

Example 2.4.8. For $(x, y, z) \in \mathbb{R}^3$ consider

$$\dot{x} = z(x + y) + x - xr^2, \quad \dot{y} = z(-x + y) + y - yr^2, \quad \dot{z} = -z|z|, \tag{2.28}$$

where $r = (x^2 + y^2)^{1/2}$ or equivalently in polar coordinates

$$\dot{r} = r - r^3, \quad \dot{\phi} = -z, \quad \dot{z} = -|z|z.$$

The global attractor is given by $\mathcal{A} = \{(x, y, z) \in \mathbb{R}^3 : z = 0, x^2 + y^2 \le 1\}$, and the dynamics on $\mathcal{A}$ is given by $\dot{r} = r - r^3$ and $\dot{\phi} = 0$. Hence, the attractor consists of the origin which is a fixed point, the circle of fixed points $\mathcal{S}^1 = \{x^2 + y^2 = 1\}$, and the radial heteroclinic connections between the origin and the points on $\mathcal{S}^1$.

However, given $z_0 \neq 0$ we obtain $z(t) = z_0/(1+|z_0|t)$ and hence $\phi(t) = \phi_0 - \operatorname{sgn}(z_0)\ln(1 + |z_0|t)$. Thus, the solution converges (algebraically slow) to $\mathcal{A}$ but it does not converge to some particular solution on $\mathcal{A}$. It can only be approximated by a sequence of solutions (i.e., fixed points on $\mathcal{S}$), with smaller and smaller errors on longer and longer time intervals. ⌋

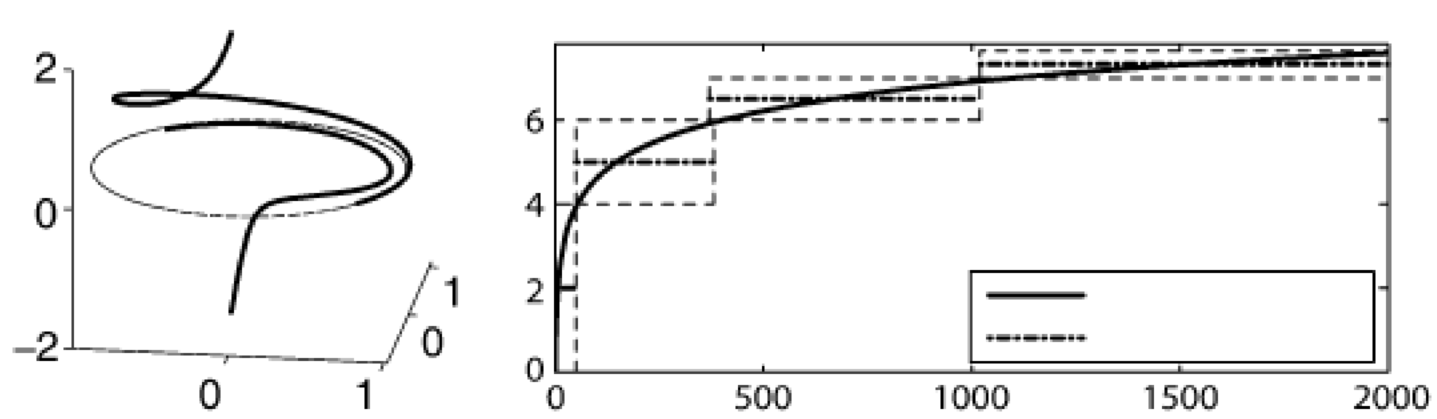


Figure 2.9. Left: Two orbits for (2.28) approaching the circle $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ of fixed points from above and below, respectively. Right: illustration of the notion of pseudo orbits for (2.28), here consisting of fixed points on S.

Another important question is the robustness of attractors under perturbations of the dynamical system. As the following example shows, in general we can only expect upper semicontinuity.

Example 2.4.9. For $0 \le \varepsilon < 1$ consider $\dot{u} = f(u, \varepsilon)$ where

$$f(u,\varepsilon) = \begin{cases} -(u+1) & \text{for } u < -2, \\ 1 - (1-\varepsilon)(u+2) & \text{for } -2 \le u < -1, \\ -\varepsilon u & \text{for } |u| \le 1, \\ -1 - (1-\varepsilon)(u-2) & \text{for } 1 < u \le 2, \\ 1 - u & \text{for } 2 < u, \end{cases}$$

cf. Figure 2.10. For all $\varepsilon > 0$ the global attractor is given by $\mathcal{A}_\varepsilon = \{0\}$. However, for $\varepsilon = 0$ we have $\mathcal{A}_0 = [-1, 1]$. ⌋

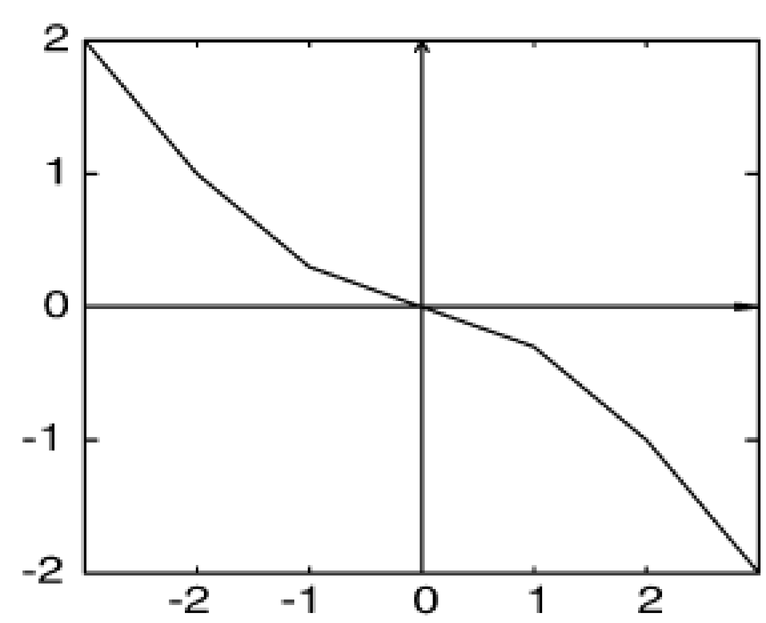


Figure 2.10. The "vector field" for Example 2.4.9 for $\varepsilon > 0$.

Theorem 2.4.10. (Attractor upper semicontinuity) *Assume that for $\mu \in [0, \mu_0)$ each of the flows $(\mathcal{S}_t^\mu)_{t\ge0}$ has a global attractor $\mathcal{A}_\mu$ such that $\cup_{0\le\mu<\mu_0}\mathcal{A}_\mu \subset Q$ for some bounded set Q, and that for each $t > 0$ the flows $\mathcal{S}_t^\mu$ converge to $\mathcal{S}_t^0$ uniformly on bounded subsets M, i.e.,*

$$\sup_{u_0\in M} \|S_t^\mu u_0 - S_t^0 u_0\|_X \to 0 \quad \text{as} \quad \mu \to 0.$$

Then

$$\operatorname{dist}(\mathcal{A}_\mu, \mathcal{A}_0) \to 0 \quad \text{as} \quad \mu \to 0.$$

Proof. Let $\varepsilon > 0$. Since $\mathcal{A}_0$ attracts Q there exists a $t > 0$ such that $\mathcal{S}_t^0 Q$ is a subset of the $\varepsilon/2$-neighborhood $N(\mathcal{A}_0, \varepsilon/2)$ of $\mathcal{A}_0$, i.e., $\mathcal{S}_t^0 Q \subset N(\mathcal{A}_0, \varepsilon/2)$. Next, for $\mu > 0$ sufficiently small, we have

$$\sup_{u\in Q} \|\mathcal{S}_t^\mu u - S_t^0 u\| \le \varepsilon/2$$

for all $u \in Q$. Since $\mathcal{A}_\mu \subset Q$ we have $\mathcal{A}_\mu = \mathcal{S}_t^\mu \mathcal{A}_\mu \subset S_t^\mu Q \subset N(\mathcal{A}_0, \varepsilon)$. □

Only with a number of additional assumptions, cf. [**Rob01**, Theorem 10.17], lower semicontinuity and hence continuity can be obtained, too. In general, as the previous example has shown, lower semicontinuity is wrong.

2.4.4. Planar systems. For autonomous ODEs in two space dimensions the possible ω-limit sets and attractors are relatively easy.

Theorem 2.4.11. (Poincaré-Bendixson) *Consider the ODE* (2.14) *in* $\mathbb{R}^2$ *and assume that the positive semiorbit* $\gamma_+(u_0)$ *through* u_0 *is bounded. If* $\omega(u_0)$ *contains no fixed point, then* $\omega(u_0)$ *is a periodic solution. If* $\omega(u_0)$ *contains a fixed point, but only finitely many, then* $\omega(u_0)$ *is either a single fixed point or it consists of fixed points with the homoclinic and heteroclinic connections between the fixed points.*

Idea of the proof. We refrain from giving a complete proof of Theorem 2.4.11, cf. [**Tes12**, §7.3], since the ideas of the proof will not be used any more in the following. The proof is based on the Jordan curve theorem saying that a closed, non-self-intersecting curve separates $\mathbb{R}^2$ in an interior and an exterior part. Since this is only true in $\mathbb{R}^2$ this assumption is essential. As a consequence, Poincaré maps have to be monotonic. See Figure 2.11.

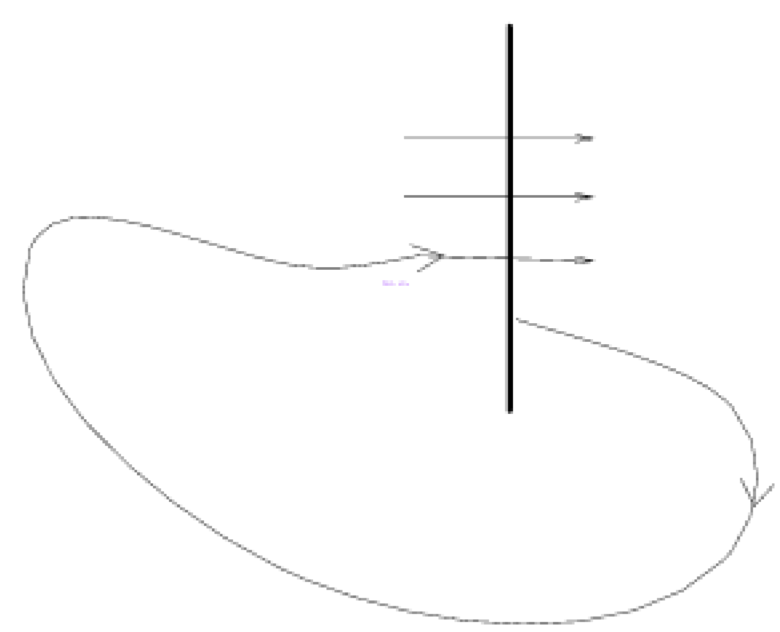


Figure 2.11. Monotonicity of the Poincaré maps.

This yields that the ω-limit set of an orbit γ intersects every Poincaré section in only one point. If there is no fixed point in $\omega(\gamma)$, then $\omega(\gamma)$ is a periodic orbit. □

Example 2.4.12. Using the Poincaré-Bendixon theorem allows us to prove the existence of a periodic solution for

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -x + y(1 - x^2 - 2y^2). \tag{2.29}$$

A direct consequence of the Poincaré-Bendixon theorem is that a positively invariant bounded set for $\dot{x}=f(x)$, $x\in\mathbb{R}^2$, which does not contain a fixed point, must contain a periodic orbit. The set $A = \{(x,y) \in \mathbb{R}^2 : 1/4 < x^2 + y^2 < 1\}$ is positively invariant for (2.29). This follows by looking at the sign of

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)^2 + y(t)^2) = 2y^2(1 - x^2 - 2y^2)$$

at the boundaries of A. We have $2y^2(1 - x^2 - 2y^2)|_{x^2+y^2=1/4} \geq 0$ and $2y^2(1 - x^2 - 2y^2)|_{x^2+y^2=1} \leq 0$. ⌋

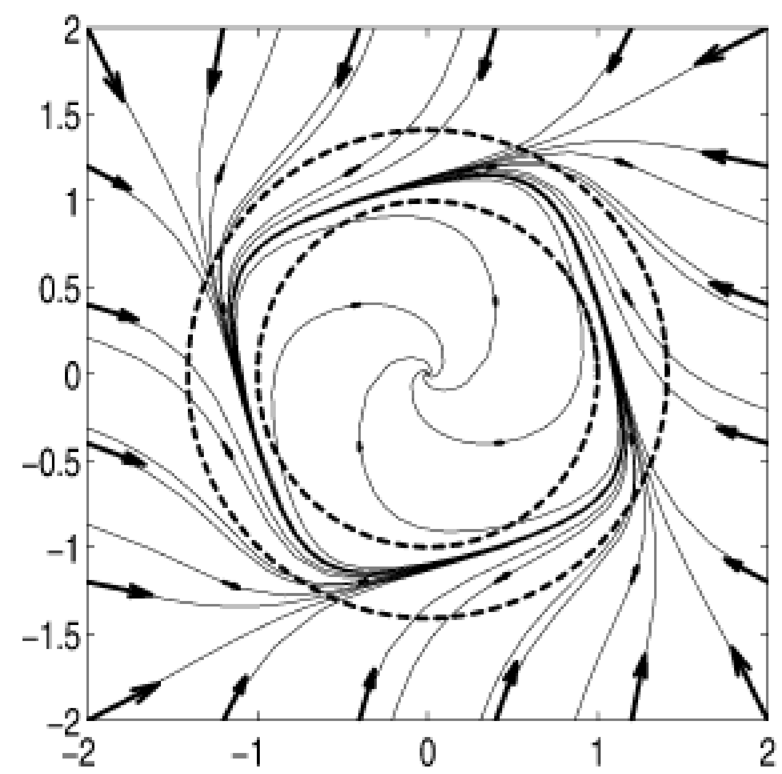


Figure 2.12. Phase portrait of (2.29) with the invariant region.

2.4.5. Gradient systems and Lyapunov functions. Another class of systems with restricted asymptotic behavior is given by gradient systems. They are of the form

$$\dot{u} = -\nabla V(u), \tag{2.30}$$

with the potential $V \in C^1(X, \mathbb{R})$, where $X = \mathbb{R}^d$ could be some general Banach space. For simplicity let $X = \mathbb{R}^d$. Obviously, fixed points of (2.30) satisfy $\nabla V(u) = 0$, i.e., they are critical points of V. Moreover, V decays along solutions of (2.30), see Figure 2.13 for a one-dimensional sketch.

Theorem 2.4.13. *The function $t \mapsto V(u(t))$ is strictly decaying for solutions $u = u(t)$ of* (2.30) *except in case that u is a fixed point. In particular, there are no non-trivial periodic solutions in gradient systems.*

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(u(t)) = (\nabla V(u(t)))^T \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t) \right) = -\|\nabla V(u(t))\|^2_{\mathbb{R}^d} < 0 \tag{2.31}$$

except in case that u is a fixed point. Suppose that there exists a non-trivial periodic solution with $u(t) = u(t+T)$ with minimal period $T > 0$. Then by (2.31) we have $V(u(t)) > V(u(t+T)) = V(u(t))$, which is a contradiction. □

Remark 2.4.14. The linearization $A \in \mathbb{R}^{d \times d}$ at a fixed point in a gradient system is a symmetric matrix. Therefore, all eigenvalues are real. ⌋

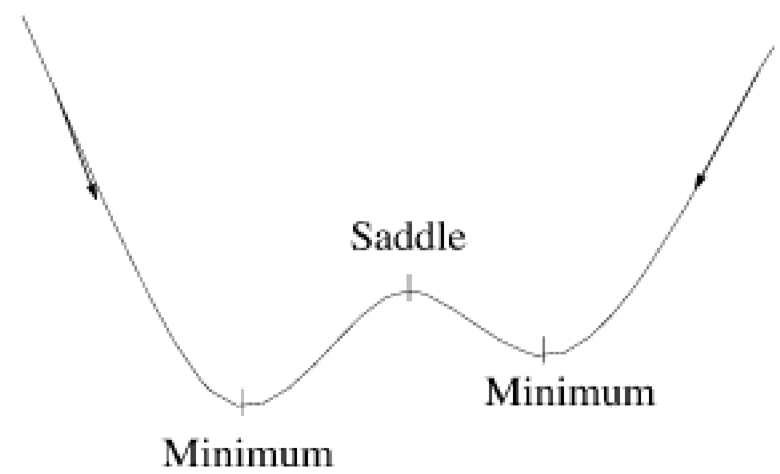


Figure 2.13. The solutions decay along the gradient of the potential.

It turns out that with only a few additional assumptions the ω-limit sets and the global attractor of gradient systems can be completely described.

Theorem 2.4.15. *Suppose that $V(u) \to \infty$ for $\|u\|_{\mathbb{R}^d} \to \infty$, that the set $\mathcal{E}$ of fixed points is finite, and that the fixed points are all hyperbolic. Then, for all $u_0 \in \mathbb{R}^d$ we have $\omega(u_0) = u^*$ for some fixed point u^*. Moreover, for* (2.30) *there exists a compact absorbing set B, and the attractor $\mathcal{A} = \omega(B)$ consists of finitely many fixed points and the connecting orbits between the fixed points.*

Proof. See [**Rob01**, §10.6.1]. □

Most properties of gradient systems are also true in case that a Lyapunov function exists for the dynamical system.

Definition 2.4.16. *A Lyapunov function for a dynamical system $\mathcal{S}_t : X \to X$ is a continuous function $\Phi : B \to \mathbb{R}$ on a positively invariant set $B \subset X$ such that*

(i) given $u_0 \in B$ the function $t \mapsto \Phi(\mathcal{S}_t u_0)$ is non-increasing,

(ii) if $\Phi(\mathcal{S}_t u_0) = \Phi(u_0)$ for some $t > 0$ then u_0 is a fixed point.

Obviously, for gradient systems $\dot{u} = -\nabla V(u)$ the potential V is a Lyapunov function on $\mathbb{R}^d$. Conversely, suppose that some dynamical system $\mathcal{S}_t$ has a Lyapunov function Φ defined on $X = \mathbb{R}^d$ with the properties that $\Phi(u) \to \infty$ as $\|u\|_{\mathbb{R}^d} \to \infty$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t)) < 0$ outside some bounded set B. For such systems we have global existence of solutions and since $\overline{B}$ is a compact absorbing set the system is dissipative. Suppose further that the set $\mathcal{E}$ of fixed points is discrete. Then the assertions of Theorem 2.4.15 remain true. Therefore, such systems are often called gradient-like. Lyapunov functions are very often used to prove stability and instability of fixed points, cf. [**HK91**, §9.3-§9.4]. See also Example 2.6.3. The concepts of gradient systems and Lyapunov functions are used in the analysis of PDEs, too, cf. §5.3.

2.5. Chaotic dynamics

In contrast to the relatively simple dynamics which can be found for autonomous ODEs in one and two dimensions, for ODEs in dimensions three and higher very complicated behavior can occur. It is reasonable to expect that complicated dynamics occurs in very high-dimensional systems such as for instance the one describing positions and velocities of 1Mol $\approx 6.022 \times 10^{23}$ particles of an ideal gas in some container. Statistical mechanics, which was initiated in 1870 by Ludwig Boltzmann, is based on the insight that a description of individual particles does not make any sense for such systems, and that a statistical description is more appropriate. However, it is surprising that very complicated dynamical behavior already occurs in low-dimensional systems. This was already observed by Henri Poincaré around 1890, when he studied the N-body problem, cf. Chapter 4. However, this fact only came apparent to a wider audience with the first computer simulations made in the early 1960s. The meteorologist Edward Lorenz [**Lor63**] found that already systems in $\mathbb{R}^3$ show a behavior which was later on called chaotic. As a consequence of the interesting pictures which were produced in the following years there was a big boom about chaos lasting for almost 30 years, cf. [**Gle88, Man91**]. According to [**Kel93**], chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions, an effect which is popularly referred to as the butterfly effect. Small differences in initial conditions (such as those due to rounding errors in numerical computation) yield widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general.

We will call a dynamical system chaotic if there is a subset of the phase space such that the flow restricted to this subset is conjugated to shift dynamics, which is a prototype of a chaotic dynamical system. The occurrence of shift dynamics in a dynamical system is proved with the help of an intermediate step which is called Smale's horseshoe. This is a geometric construction of a chaotic dynamical system which is easier to detect in a given dynamical system. Using this idea we will present with Silnikov chaos an example of a three-dimensional ODE which exhibits chaotic behavior. Routes to chaos in dissipative systems by sequences of local bifurcations will briefly be described subsequently in §3.4. The occurrence of chaotic behavior in Hamiltonian systems is discussed in §4.5. In the present book, chaos will not play a central role, but one should keep in mind its existence already in low-dimensional dynamical systems. Our presentation of this subject is based on [**GH83**].

2.5.1. Shift dynamics. We will use shift dynamics as a prototype of a chaotic dynamical system. On the set

$$\Sigma_2 = \{a : \mathbb{Z} \to \{0,1\} : a = (a_i)_{i\in\mathbb{Z}}\}$$

which is equipped with the distance

$$d(a,b) = \sum_{j\in\mathbb{Z}} 2^{-|j|}|a_j - b_j|.$$

we define the shift $\sigma : \Sigma_2 \to \Sigma_2$ by $(\sigma(a))_i = a_{i+1}$. Similarly, we define Σ_2^+ where the index set $\mathbb{Z}$ is replaced by $\mathbb{N}$. The subsequent theory will be formulated for the shift in Σ_2, but can also be formulated in Σ_2^+.

Theorem 2.5.1. *We have that $\sigma \in C(\Sigma_2, \Sigma_2)$ has the following properties:*

(i) There exist non-trivial periodic solutions to every minimal period;

(ii) There exists a dense orbit;

(iii) The sensitivity w.r.t. the initial conditions holds, i.e., for every $a \in \Sigma_2$ and every $\delta > 0$ there exist $b \in \Sigma_2$ and $j \geq 0$ such that $d(\sigma^j(a), \sigma^j(b)) \geq 1$, although $d(a,b) \leq \delta$.

Proof. (i) The 1-periodic solutions are $a = \dots 00000 \dots$ and $a = \dots 11111 \dots$. The 2-periodic solutions are generated by 00, 01, 10 and 11, and the 3-periodic ones by 000, 001, 010, 100, 110, 101, 011 and 111, etc.
(ii) Consider the orbit to the initial condition a, consisting of all finite sequences that generate the periodic solutions, i.e.,

$$a = \dots 0000|0100011011000001010100110101011111000000010010\dots.$$

Right of | we have the position $j = 0$ and left of | the sequence is filled up with zeroes. For a given $\varepsilon > 0$ and $b \in \Sigma_2$ we have to find $n \in \mathbb{N}$ such that $d(b, \sigma^n(a)) \leq \varepsilon$. For $c \in \Sigma_2$ we have $d(b,c) \leq \varepsilon$, if at least $b_j = c_j$ for $|j| \leq j_0(\varepsilon)$. The other c_j for $|j| \geq j_0(\varepsilon)$ can be arbitrary. Since a contains all finite sequences the claim follows by shifting a until the finite sequence $(b_j)_{j=-j_0,\dots,j_0}$ occurs at the positions between $-j_0$ and j_0.
(iii) Let $a \in \Sigma_2$ and $\delta > 0$, and set $b_j = a_j$ for $j \leq j_0(\delta)$ and $a_{j_0(\delta)+1} \neq b_{j_0(\delta)+1}$. Then $d(a,b) \leq \delta$, but $d(\sigma^{j_0(\delta)+1}(a), \sigma^{j_0(\delta)+1}(b)) \geq 1$. □

A general dynamical system is called chaotic if the flow is conjugated to shift dynamics on a subset of its phase space, i.e.,

Definition 2.5.2. *A discrete dynamical system $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ is called chaotic, if there is a set $\Lambda \subset \mathbb{R}^d$ and a homeomorphism $h : \Lambda \to \Sigma_2$ such that on Λ the flows are conjugated, i.e., $\sigma \circ h|_\Lambda = h \circ \Pi|_\Lambda$.*

Remark 2.5.3. This is a very strict definition of a chaotic dynamical system. Chaos in the sense of [**Dev89**] for a map $f \in C(M,M)$, with M some metric space, is defined by

(1) the sensitive dependence on initial conditions;

(2) periodic points are dense in M;

(3) topological transitivity, i.e., for all open subsets U, V of M, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. ⌋

Remark 2.5.4. Another definition of chaotic behavior in a system where all forward orbits are bounded is the occurrence of a positive Lyapunov exponent [**Rue89**]. Lyapunov exponents describe how the distance of nearby solutions evolves in time. They are defined through

$$\lambda(u_0, \varphi) = \limsup_{t \to \infty} \left(\frac{1}{t} \ln \|Du(t, u_0)\varphi\| \right)$$

for $\varphi, u_0 \in \mathbb{R}^d$. For each initial condition u_0 there are d such Lyapunov exponents. Very often the Lyapunov exponents do not depend on u_0. A positive Lyapunov exponent implies a sensitive dependence on the initial conditions. ⌋

See Exercise 2.18 for an example of a map where conjugacy to the shift on Σ_2^+ can be shown explicitly. A related and famous 1D iteration for which chaotic behaviour can be shown for certain parameters is the logistic map, see, e.g., [**Dev89**], and §3.4.1.

2.5.2. Smale's horseshoe. The occurrence of shift dynamics in a general dynamical system is very often proved with the help of an intermediate step. There is a geometric construction of a chaotic dynamical system, called Smale's horseshoe, which is easier to detect in a given dynamical system than shift dynamics. The construction is as follows. Starting with $S = [0,1] \times [0,1] \subset \mathbb{R}^2$ we define via Figure 2.14 a map $f : S \to \mathbb{R}^2$ such that $f(S) \cap S$ consists of two components, namely the two vertical strips V_0 and V_1. There exist two horizontal strips H_0 and H_1 with $f(H_j) = V_j$.

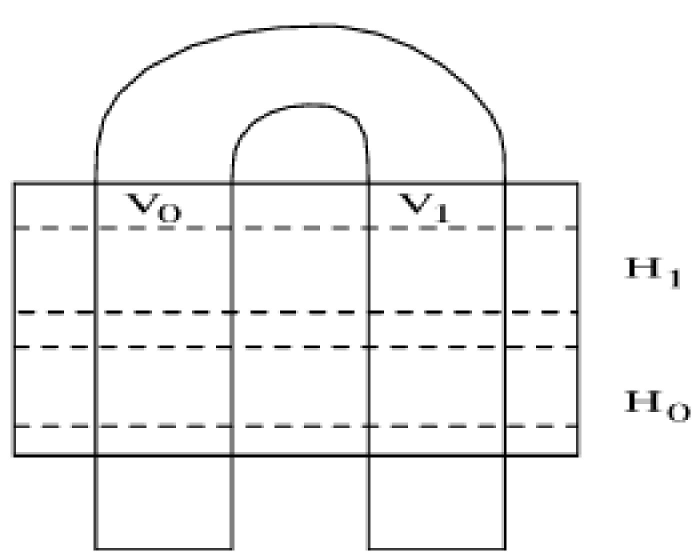


Figure 2.14. Smale's horseshoe.

Under iteration of f most of the points of S leave S. The points which stay in S under all iterations of f define a set

$$\Lambda = \{x : f^i(x) \in S,\ -\infty < i < \infty\}.$$

Remark 2.5.5. The invariant set $\Lambda \subset \mathbb{R}^d$ is called hyperbolic since there is a continuous invariant splitting of the tangent spaces $T_\Lambda \mathbb{R}^d = E^u_\Lambda \oplus E^s_\Lambda$ with the following property: There exist constants $C > 0$ and $\lambda \in (0,1)$ with

$$\begin{aligned} &\text{a)} \quad \|Df^{-n}(x)v\|_{\mathbb{R}^d} \le C\lambda^n \|v\|_{\mathbb{R}^d} \text{ if } v \in E^u_\Lambda(x), \\ &\text{b)} \quad \|Df^{n}(x)v\|_{\mathbb{R}^d} \le C\lambda^n \|v\|_{\mathbb{R}^d}, \text{ if } v \in E^s_\Lambda(x). \end{aligned}$$

Hyperbolic dynamics, cf. [**KH97**, Part 4], is one branch in the description of chaos. ⌋

The set Λ has a complicated topological structure.

Lemma 2.5.6. *The set Λ is a Cantor set, i.e., an uncountable, compact, totally disconnected and nowhere dense set which consists entirely of limit points.*

Proof. Each horizontal strip H_i is mapped through f into the vertical strip $V_i = f(H_i)$. We consider $V_i \cap H_j$ which is the image of some thin horizontal strips H_{ij}. We obtain vertical strips $V_{ij} = f^2(H_{ij})$ by two iterations of f. See Figure 2.15.

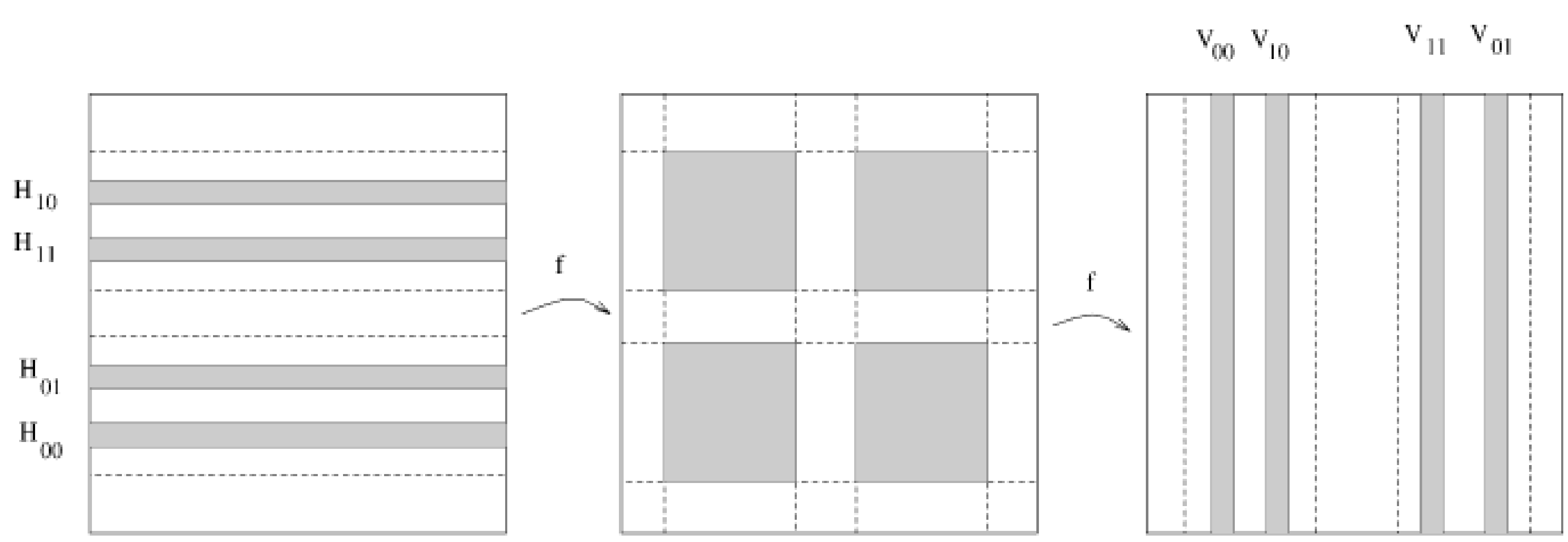


Figure 2.15. Iteration of the horseshoe map f.

By more forward and backward iterations of f and the intersection of all horizontal and vertical strips which are obtained in this way we get a closed, non-empty, completely disconnected set Λ. Each point in Λ is a limit point of Λ. Thus, Λ is a Cantor set. $\square$

To each point $x \in \Lambda$ we associate an infinite sequence $a : \mathbb{Z} \to \{0,1\}$ via $\phi(x) = (a_i)_{i=-\infty}^{\infty}$ if $f^i(x) \in H_{a_i}$.

Theorem 2.5.7. *There is a one-to-one map ϕ between Λ and Σ_2 such that the sequence $b = \phi(f(x))$ can be computed from $a = \phi(x)$ by shifting the indices $b_i = a_{i+1}$. The map ϕ is a homeomorphism between the metric spaces (Σ_2, d) and $(\Lambda, \|\cdot\|_{\mathbb{R}^d})$.*

Proof. We consider $\phi(f(x)) = (b_i)_{i=-\infty}^{\infty}$ with $f^{i+1}(x) \in H_{b_i}$. Hence, $f^i(x) \in H_{b_{i-1}} = H_{a_i}$ and therefore $b_i = a_{i+1}$ which implies $\phi \circ f = \sigma \circ \phi$.

It remains to prove the continuity of $\phi : \Lambda \to \Sigma_2$ and $\phi^{-1} : \Sigma_2 \to \Lambda$. For $x \in \Lambda$ and all $\varepsilon > 0$ we have to show the existence of a $\delta > 0$ such that $d(\phi(x), \phi(y)) < \varepsilon$ if $\|x - y\| < \delta$. To a given $\varepsilon > 0$ there exists a $j_0 = j_0(\varepsilon)$ such that $d(a, b) \leq \varepsilon$ is equivalent to $a_j = b_j$ at least for $|j| \leq j_0$. a_j and b_j can be arbitrary for $|j| > j_0$. Therefore, the condition $d(\phi(x), \phi(y)) < \varepsilon$ uniquely defines two sequences $\overline{a}^+ = (a_i)_{j=0}^{j_0}$ and $\overline{a}^- = (a_i)_{j=-j_0-1}^{-1}$. Associated with these sequences there are strips $V_{\overline{a}^-}$ and $H_{\overline{a}^+}$. If we choose $\delta > 0$ so small that $y \in V_{\overline{a}^+} \cap H_{\overline{a}^-}$ if $\|y - x\|_{\mathbb{R}^d} \leq \delta$ we are done. The continuity of ϕ^{-1} follows in the same way. $\square$

As a consequence it follows that if we find a Smale's horseshoe in a dynamical system, then chaotic behavior in the sense of Definition 2.5.2 is present in this system.

2.5.3. Silnikov chaos. Using the idea of Smale's horseshoe we present a first example of a chaotic dynamical system coming from an ODE, namely Silnikov chaos. The presentation is based on [**GH83**, §6.5.1]. However, we skip most analytic arguments and argue mostly by pictures. We consider an autonomous three-dimensional ODE with a homoclinic orbit γ at the origin which is assumed to be a hyperbolic fixed point with eigenvalues $\lambda \in \mathbb{R}, \omega, \overline{\omega} \in \mathbb{C}$, where $\operatorname{Im}\omega \neq 0$. See Figure 2.16(a). Silnikov [**Sil65**] proved in 1965 the following result.

Theorem 2.5.8. *If $|\operatorname{Re}\omega| < \lambda$, then the flow S_t can be perturbed in such a way that the perturbed flow $\widetilde{S}_t$ has a homoclinic orbit $\widetilde{\gamma}$ close to γ and that there exists a subset of $\mathbb{R}^3$ on which the first return map for the perturbed flow $\widetilde{S}_t$ is conjugated to Smale's horseshoe.*

Idea of the proof. By the Hartman-Grobman theorem 2.3.8 we may assume that the vector field is linear in a neighborhood of the origin, i.e., with $\alpha := \operatorname{Re}\omega$,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \omega = \alpha + \mathrm{i}\beta. \tag{2.32}$$

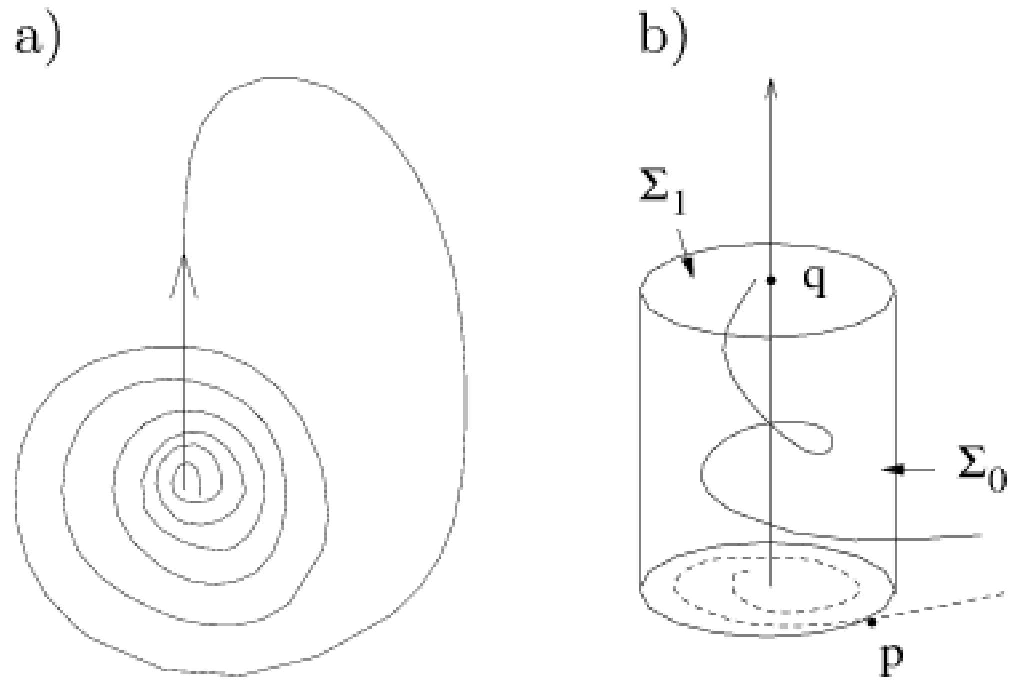


Figure 2.16. a) The homoclinic orbit in the example of Silnikov, b) the inner map $\psi_{int} : \Sigma_0 \to \Sigma_1$, and c) the outer map ψ_{out} : $\Sigma_1 \to \Sigma_0$.

The solutions of (2.32) are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} (t) = \begin{pmatrix} e^{\alpha t}((\cos \beta t)x(0) - (\sin \beta t)y(0)) \\ e^{\alpha t}((\sin \beta t)x(0) + (\cos \beta t)y(0)) \\ e^{\lambda t}z(0) \end{pmatrix}. \tag{2.33}$$

We define two sets

$$\begin{aligned} \Sigma_0 =& \{(x, y, z) : x^2 + y^2 = r_0^2 \text{ and } 0 < z < z_1\}, \\ \Sigma_1 =& \{(x, y, z) : x^2 + y^2 < r_0^2 \text{ and } z = z_1 > 0\} \end{aligned}$$

and assume that these sets are contained in the previous neighborhood. The solutions go from Σ_0 to Σ_1 according to Figure 2.16 b). The inner map $\psi_{int} : \Sigma_0 \to \Sigma_1$, which maps a point $a \in \Sigma_0$ into the first intersection point of the associated solution with Σ_1, maps vertical vertical lines from Σ_0 into a logarithmic spiral in Σ_1. The outer map ψ_{out} transports a neighborhood of q through the homoclinic solution into

$$\widetilde{\Sigma}_0 = \{(x, y, z) : x^2 + y^2 = r_0^2,\ |z| < z_1\ \}.$$

See Figure 2.16 c). The map ψ is defined by $\psi = \psi_{out} \circ \psi_{int}$ for all points $X \in \Sigma_0$ with $\psi(X) \in \Sigma_0$ and has the same asymptotic behavior as ψ_{int} for $z \to 0$ since for $z \to 0$ the time needed by the solution to come from Σ_0 to Σ_1 becomes infinite, whereas the time needed by the solution to come from Σ_1 to Σ_0 stays finite. Hence, a rectangular set R around the entrance point p of the homoclinic orbit is mapped into a spiral like structure. See Figure 2.17. The assumption $|\operatorname{Re}\omega| < \lambda$ is necessary that this picture really occurs, for more details see [**GH83**, §6.5.1]. Therefore, graphically we have found a Smale's horseshoe for ψ. $\square$

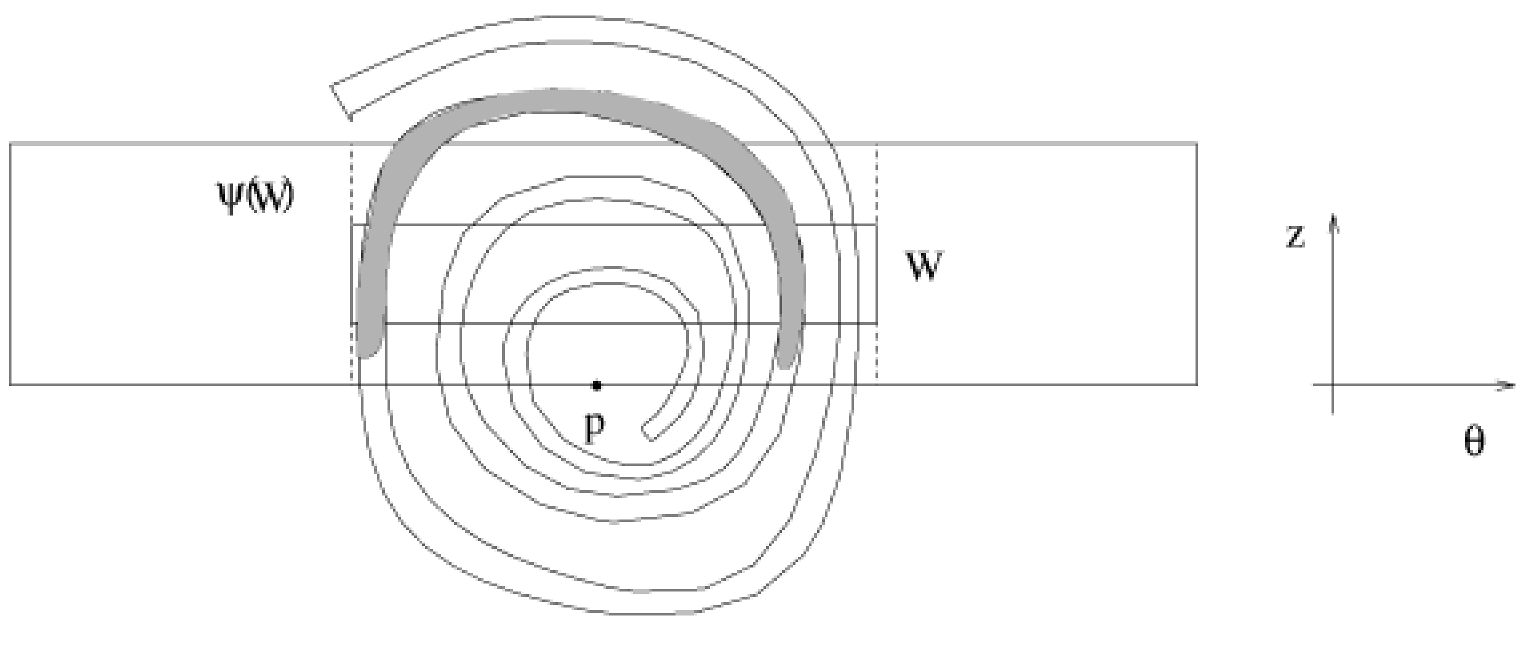


Figure 2.17. Smale's horseshoe in the Silnikov example.

2.6. Examples

The following series of examples is intended to give some familiarity with the notions and ideas introduced so far.

Example 2.6.1. a) For the potential $V(x,y) = (x^2-1)^2 + y^2$ we find $-\nabla V(x,y) = -(4x(x^2-1), 2y)$, leading to the fixed point $(x,y) = (0,0)$, which is a saddle point of V, and to the fixed points $(\pm 1, 0)$, which are minima of V. For every $r > 1$ the set $\{(x,y) : x^2 + y^2 \le r^2\}$ is absorbing. The attractor $\mathcal{A}$ is given by $[-1,1] \times \{0\}$, consisting of the three fixed points and the heteroclinic connections between the unstable fixed point $(0,0)$ and the stable fixed points $(\pm 1, 0)$.

b) For the potential $V(x,y) = (x^2-1)^2 + (y^2-1)^2$ we find that any neighborhood of the unit square $Q = [-1,1] \times [-1,1]$ is an absorbing set. The global attractor is $Q = \overline{W^u((0,0))}$. ⌋

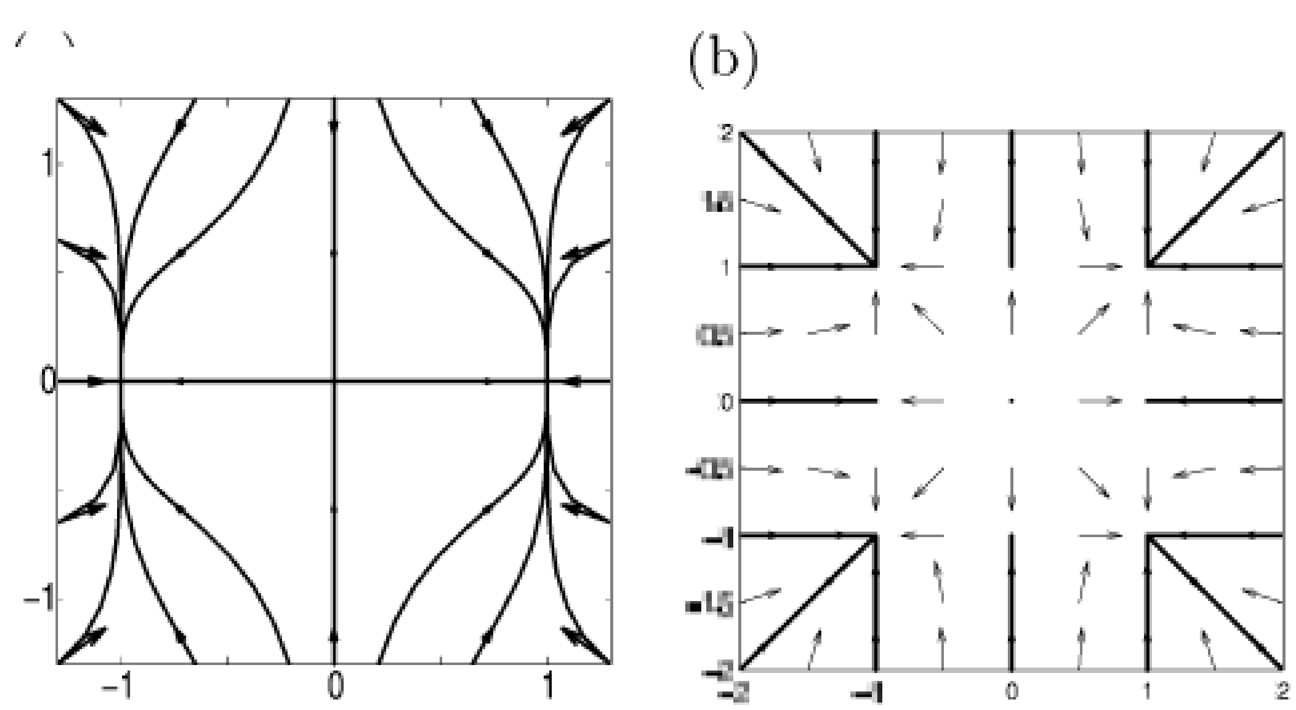


Figure 2.18. Phase portraits for Example 2.6.1 a) and b).

Example 2.6.2. We consider

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -cy + x - x^3.$$

There is a simple mechanical interpretation of the orbits of this system as the orbits of a particle moving in the double-well potential $F(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ with friction $-cy$. In partiular, for $c = 0$ we have a similar situation as in Example 2.3.20, i.e., orbits are level lines of the energy $E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + F(x)$, see also Example 4.1.2 below for a general discussion.

The system possesses the fixed points $(x, y) = (\pm 1, 0)$ and $(x, y) = (0, 0)$. The linearization at the fixed point $(x, y) = (0, 0)$ yields the eigenvalues $\lambda_{1,2} = -c/2 \pm \sqrt{c^2/4 + 1}$. Hence, $(0, 0)$ is a saddle for all values of c. The unstable eigenspace is spanned by $\phi_1 = (1, 1)$, the stable one by $\phi_2 = (1, -1)$. The linearization at the fixed point $(x, y) = (1, 0)$ yields the eigenvalues $\lambda_{1,2} = -c/2 \pm \sqrt{c^2/4 - 1}$. Thus, $(1, 0)$ is a center for $c = 0$ (Figure 2.19(a)), a stable vortex for $0 < c < 2$ (Figure 2.19(b)), a stable node for $c \geq 2$ (Figure 2.19(c)), an unstable vortex for $-2 < c < 0$, and an unstable node for $c \leq -2$. The same classification holds for the fixed point $(-1, 0)$. The mechanical interpretation is that for $c > 2$ the friction is so large that the particle approaches the minima $x = \pm 1$ of the energy monotonically.

For $c > 0$ the system is dissipative, and the stable manifold $W_s((0, 0))$ separates the stable manifolds $W_s((1, 0))$ and $W_s((-1, 0))$, i.e., the domains of attraction of $(1, 0)$ and $(-1, 0)$. The global attractor consists of the three fixed points and the unstable manifold of $(0, 0)$.

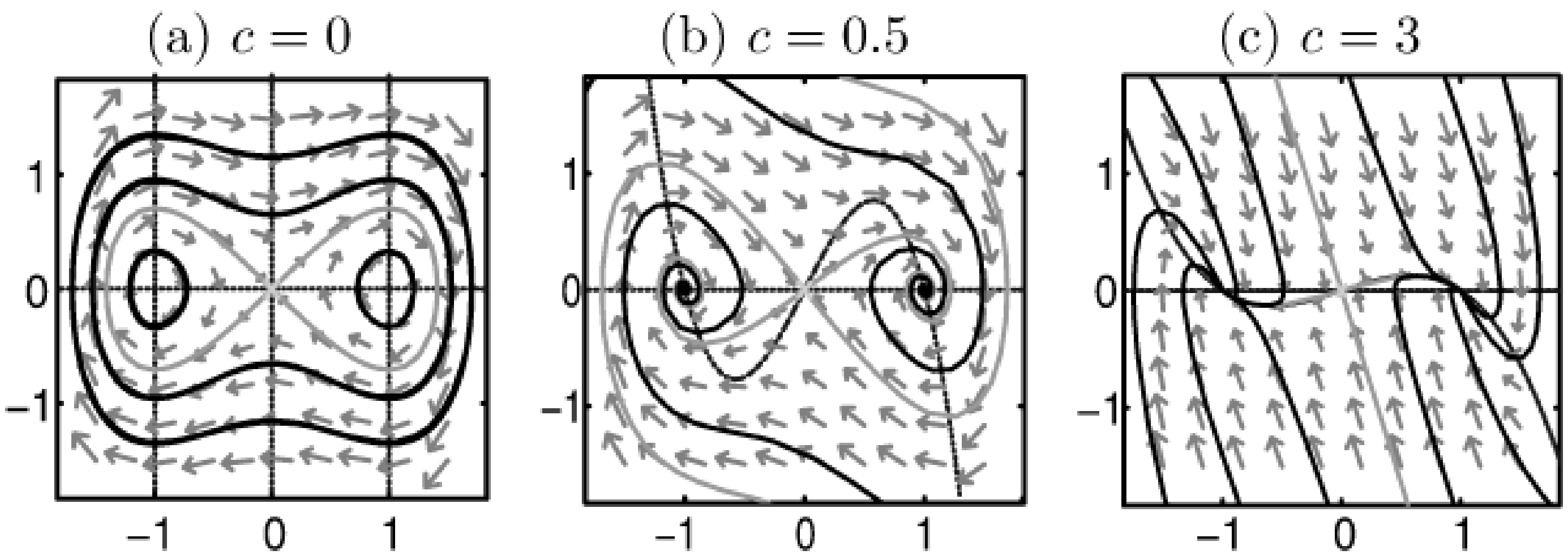


Figure 2.19. Phase portraits for Example 2.6.2; invariant manifolds in light grey, and nullclines as dashed lines.

For $c = 0$ we have homoclinic solutions and so $W_u((0, 0)) = W_s((0, 0))$, and the center manifolds of $(\pm 1, 0)$ can be defined as small disks around $(\pm 1, 0)$. To show the existence of the homoclinic solutions for $c = 0$, instead of the energy argument that $E(x, \dot{x})$ =const= 0 we may also use the symmetry (reversibility) $(t, x, y) \mapsto (-t, x, -y)$ under which the system is invariant. Hence, with $t \mapsto (x(t), y(t))$ also $t \mapsto (x(-t), -y(-t))$ is a solution. The unstable manifold of the origin intersects the fixed space $\Sigma = \{(x, 0) : x \in \mathbb{R}\}$ of reversibility transversally. W.l.o.g. taking this intersection at $t = 0$, the orbit can be extended to $t > 0$ by reflection at Σ. See Figure 2.20.

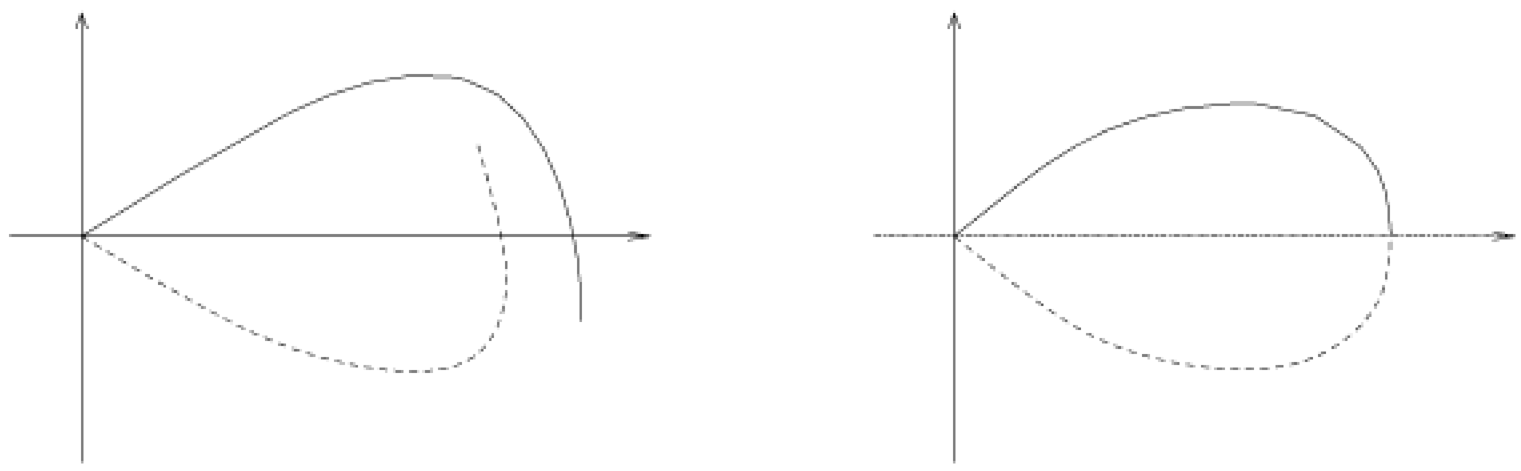


Figure 2.20. Persistence of homoclinic connections in planar systems. In non-reversible systems an additional parameter is needed for the intersection of the stable and unstable manifold. In reversible systems the fact that the stable manifold intersects the fixed space $\{y = 0\}$ of the reversibility operator transversally can be used for the persistence.

This homoclinic orbit persists under small perturbations respecting the reversibility of the vector field due to the transversal intersection. Such transversibility and symmetry arguments often also work when other arguments, such as the above energy argument, fail, see, e.g., Remark 13.3.1. For general perturbations ($c > 0$) the homoclinic orbit breaks up in accordance with the fact that the probability that two one-dimensional manifolds intersect in a two-dimensional phase space is zero. ⌋

The following examples are applications from mathematical biology; examples of this type, combined with diffusion lead to the important class of reaction diffusion systems, see Chapter 9.

Example 2.6.3. (Lotka-Volterra) We consider the predator-prey system

$$\dot{x} = x(a - y), \ \dot{y} = y(x - c), \quad a, c > 0 \tag{2.34}$$

with $x, y > 0$ ($x =$ prey, $y =$ predator). In case of no predators, i.e., $y = 0$, the prey will grow with some exponential rate according to $\dot{x} = ax$. In case of no prey, i.e., $x = 0$, the predators will die with some exponential rate according to $\dot{y} = -cy$. In case of predators, i.e., $y > 0$, the prey will be killed by the predator via the term $-xy$ with a rate proportional to the number of predators. On the other hand the term xy gives an exponential growth of the number of predators with a rate proportional to the number of preys.

The unique non-trivial fixed point is $(x_0, y_0) = (c, a)$. Its linearization

$$A = \begin{pmatrix} a - y & -x \\ y & x - c \end{pmatrix}\Bigg|_{(x,y)=(c,a)} = \begin{pmatrix} 0 & -c \\ \gamma & 0 \end{pmatrix},$$

possesses the eigenvalues $\lambda = \pm \mathrm{i}\sqrt{ca}$ such that (x_0, y_0) is a centre for the linearization. Thus, no stability result can be concluded from Theorem 2.3.4 a) for the nonlinear system.

However, $\phi(x, y) = x + y - c\ln(x) - a\ln(y)$ is conserved for (2.34), i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\phi(x, y) = 0$. Since $(\dot{x}, \dot{y}) \neq (0, 0)$ for $(x, y) \neq (x_0, y_0)$ the solutions move on the level lines of ϕ. Calculus yields that ϕ has a unique critical point, namely a minimum in (x_0, y_0). Thus, (x_0, y_0) is nonlinearly stable and all other solutions move on periodic orbits around (x_0, y_0). This behavior agrees with observed data, for instance of canadian lynx and snowshoe hare pelt-trading records of the Hudson Bay Company between 1845 and 1935, cf. [**Mur89**].

The Lotka–Volterra model (2.34) as the oldest predator-prey model was partially motivated by the observation that during and shortly after world war I the fraction of predator fish caught in the mediterranean sea increased, when the total fishing decreased. Let $(x, y)(t)$ be a periodic solution of (2.34) with period T. Then

$$\overline{x} = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t = c, \qquad \overline{y} = \frac{1}{T}\int_0^T y(t)\,\mathrm{d}t = a.$$

This holds due to $\frac{1}{T}\int_0^T \frac{\dot{x}}{x}\,\mathrm{d}t = \frac{1}{T}\int_0^T a - y\,\mathrm{d}t = \frac{1}{T}(\ln x(T) - \ln x(0)) = 0$, hence $\overline{y} = a$, and similarly for $\overline{x} = c$. Fishing can be modeled by a simple modification of (2.34), namely to replace a, c by $a - p$ and $c + p$, where $0 \leq p \leq a$ denotes the "fishing pressure". Depending on p, the fraction of predators hence is

$$\frac{\overline{y}(p)}{\overline{x}(p) + \overline{y}(p)} = \frac{a - p}{a + c},$$

which is a decreasing function of p.

See also Exercise 2.19 for some modification of (2.34), for which the conserved quantity ϕ becomes a genuine Lyapunov function. $\lrcorner$

Example 2.6.4. (SI and SIS diseases) We are interested in the dynamics of some disease which proceeds on a time scale much shorter than the lifespan of its hosts. Thus, we assume that the size of the population is unchanged and that in the population of size N a fraction S of individuals is healthy, but susceptible to this disease, while a fraction I is infected. A general ODE model describing the evolution of the fractions S and I reads

$$\dot{S} = -f(S, I), \qquad \dot{I} = f(S, I),$$

where $f(S, I)$ is the rate of infections. The simplest model is $f(S, I) = \beta IS$, which in chemistry would be called the law of mass action, see also §9.1. Here β is called the pairwise infectious contact rate. Using $S + I = N$ yields

$$\dot{I} = \beta I(N - I).$$

It is easy to see that in this model with $\beta > 0$ the whole population becomes infected.

A slight extension is given by SIS models, where infected recover and become susceptible again with a rate γ. Thus

$$\dot{S} = -f(S, I) + \gamma I, \qquad \dot{I} = f(S, I) - \gamma I.$$

Introducing dimensionless variables $u = S/N, v = I/N, \tau = \gamma t$, and using $u + v = 1$ we obtain

$$v' = (R_0(1 - v) - 1)v,$$

where $R_0 = \beta N/\gamma$ is called the reproductive ratio of the disease and where $'$ denotes the derivative w.r.t. the new time variable τ. For $R_0 < 1$ the disease dies out, but for $R_0 > 1$ it becomes endemic, i.e., it reaches the steady state $1 - 1/R_0$ as $t \to \infty$. Vaccination reduces the number of susceptible and hence R_0. Note that for desease control it is not necessary to vaccinate all, but sufficiently many to decrease R_0 below 1. ⌋

Example 2.6.5. (Mathematical ecology) The Kolmogorov form of the equations for 2-species interaction in mathematical ecology is

$$\dot{u} = uM(u, v), \qquad \dot{v} = vN(u, v), \tag{2.35}$$

where $(u, v) = (u, v)(x, t)$ are population densities and their respective growth rates M and N are smooth functions from $\mathbb{R}^2_+$ to $\mathbb{R}$. The models (2.35) are further classified as

$$\begin{aligned} \text{predator-prey (PP)} &\quad \partial_v M < 0 \text{ and } \partial_u N > 0 \text{ for } u, v > 0,\\ \text{competition (C)} &\quad \partial_v M < 0 \text{ and } \partial_u N < 0 \text{ for } u, v > 0,\\ \text{symbiosis (S)} &\quad \partial_v M > 0 \text{ and } \partial_u N > 0 \text{ for } u, v > 0. \end{aligned}$$

Usually, further conditions are imposed, namely

(PP1) $\exists k_0 > 0$ such that $M(u, 0) < 0$ for $u > k_0$,

(PP2) $\exists$ a function l such that $N(u, v) < 0$ for $u > 0$ and $v > l(u)$.

(C1) $\exists k_0 > 0$ such that $M(u, 0) < 0$ for $u > k_0$,

(C2) $\exists l_0 > 0$ such that $N(u, v) < 0$ for $v > l_0$.

(S1) $\exists$ a function k such that $M(u, 0) < 0$ for $u > k(v)$,

(S2) $\exists$ a function l such that $N(u, v) < 0$ for $v > l(u)$,

(S3) $k(v) = o(v)_{v\to\infty}$ and $l(u) = o(u)_{u\to\infty}$.

Biologically, for instance (PP1) and (C1) essentially mean that even if there are no predators ($v = 0$), then the growth of the prey population saturates at k_0. By (PP2) and (C2), the predators saturate at $l(u)$. Finally, symbiosis means that each species thrives with the other, but (S3) ensures limits to this symbiotic growth. The condition (S3) should be complemented by demanding that there exists at least one non-trivial fixed point.

As examples consider

$$\begin{aligned}
&(1) \quad \dot{u} = u(1-u-v), \quad \dot{v} = v(u-v),\\
&(2) \quad \dot{u} = u(1-u-v), \quad \dot{v} = v(1-u-\alpha v),\\
&(3) \quad \dot{u} = u(2\arctan(2v)-u), \quad \dot{v} = v(3\arctan(2u)-v).
\end{aligned}$$

In (2), $\alpha > 0$ is some parameter. Clearly, (1)=(PP), (2)=(C), (3)=(S). After determining the unique fixed points $(u,v)^*$ with $uv > 0$ for (1) and (3), the phase portraits can be conveniently sketched by considering the signs of the growth rates M, N. For (1) we may additionally use the fact that, e.g., $[1/4, 3/4] \times [1/4, 3/4]$ is positively invariant.

For (2) we note that for $\alpha = 1$ we have $M = N$, and thus a line $\{u+v = 1\}$ of fixed points. For $\alpha \neq 1$ we again have a unique non-trivial fixed point. In particular, for $\alpha > 1$ ($\alpha < 1$) the v species (the u species) dies out. For $\alpha > 1$ the biological interpretation is that for $u = v$ the growth rate N of v is smaller than that of u, due to higher damping (faster saturation) of the growth of v by itself, hence u "wins". ⌋

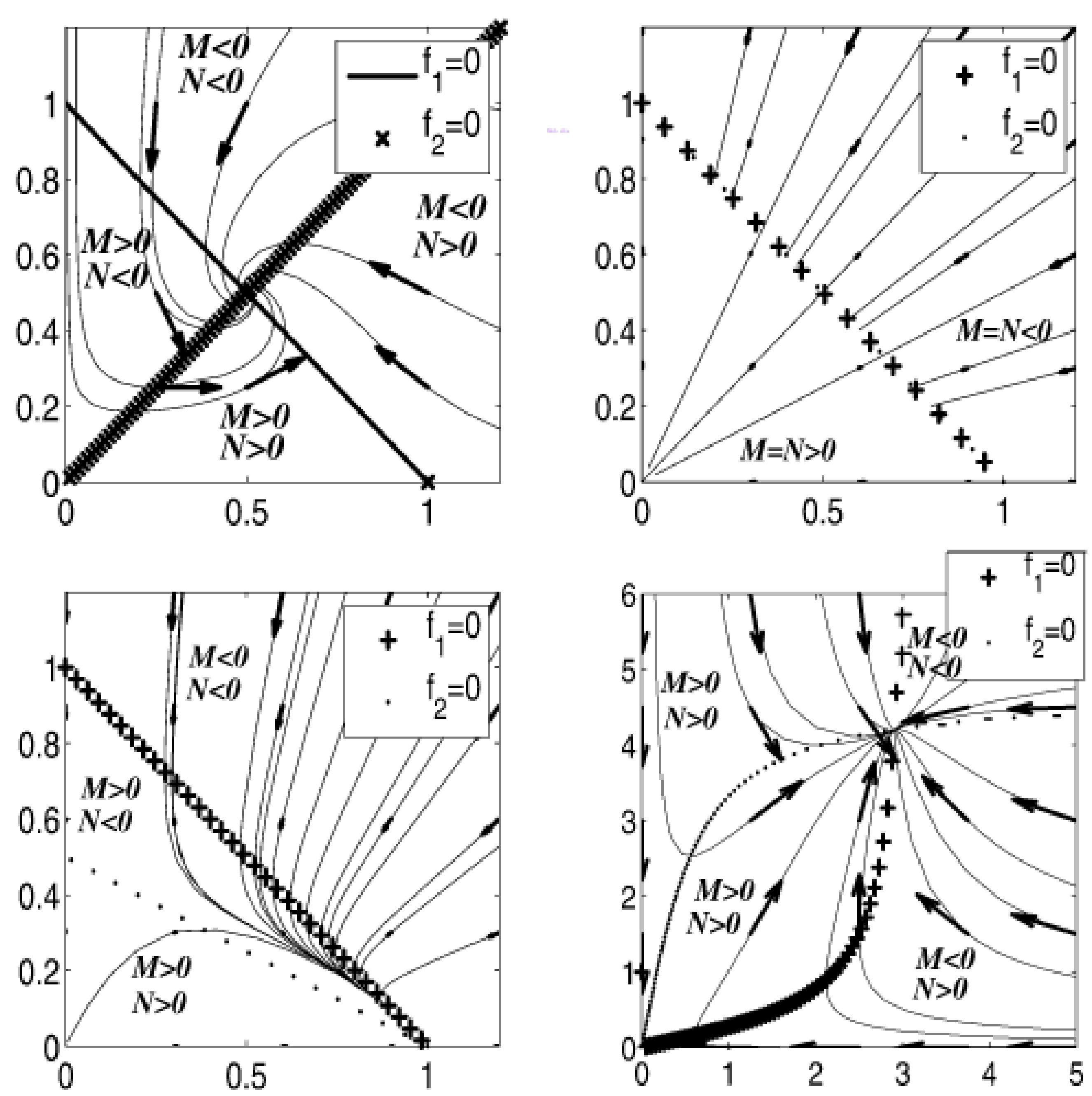


Figure 2.21. ODE phase portraits for (PP), (C) ($\alpha = 1$ and $\alpha = 2$) and (S). For $\alpha = 1$ in (C) we have a line $\{u+v=1\}$ of fixed points. For equations of the form (2.35) a convenient way to start the phase portrait is to consider the regions defined by the nullclines, i.e., $M = 0$ and $N = 0$.

Example 2.6.6. The van der Pol equation [**VdP26**] is given by

$$\ddot{u} + \varepsilon(u^2 - \alpha)\dot{u} + u = 0, \quad u(t) \in \mathbb{R}, \tag{2.36}$$

where $\alpha > 0$ and $0 \leq \varepsilon \ll 1$ are some parameters. As initial conditions we take $u(0) = a$ and $\dot{u}(0) = 0$. This describes some oscillator with small amplitude-dependent damping. It is known and might be expected from the form of the equation, that for every fixed $\alpha > 0$ and small $\varepsilon > 0$ there is a unique periodic solution. For $\varepsilon = 0$ we have solutions $u(t) = A\mathrm{e}^{\mathrm{i}t} + \text{c.c.}$ with $A \in \mathbb{C}$ arbitrary, and thus for $\varepsilon > 0$ we try a two-scale ansatz of the form

$$u(t) = A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}. \tag{2.37}$$

Using $u^2 = A^2\mathrm{e}^{2\mathrm{i}\omega t} + 2|A|^2 + \overline{A}^2\mathrm{e}^{-2\mathrm{i}\omega t}$ this yields

$$\begin{aligned}
&\mathcal{O}(\varepsilon^0): \quad -\omega^2 + 1 = 0, \quad \Rightarrow \omega = 1,\\
&\mathcal{O}(\varepsilon^1): \quad 0 = \mathrm{i}(-2\frac{\mathrm{d}}{\mathrm{d}\tau}A + \alpha A - A|A|^2)\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A^3\mathrm{e}^{3\mathrm{i}t} + \text{c.c.},
\end{aligned}$$

and thus equating the coefficient of $\varepsilon^1\mathrm{e}^{\mathrm{i}t}$ to zero yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A = \frac{1}{2}(\alpha A - A|A|^2), \tag{2.38}$$

which is called the Landau equation for this problem. Introducing polar coordinates $A(\tau) = \rho(\tau)\mathrm{e}^{\mathrm{i}\phi(\tau)}$ gives $\rho' = \frac{1}{2}\rho(\alpha - \rho^2)$, $\phi' = 0$, with the abbreviation $' = \frac{\mathrm{d}}{\mathrm{d}\tau}$. From this, or directly from (2.38), we can see that $|A|$ converges to $\sqrt{\alpha}$, which predicts that u approaches the circle with radius $2\sqrt{\alpha}$ up to $\mathcal{O}(\varepsilon)$ terms. Incidentically, although nonlinear, (2.38) can be explicitly solved. Via $r = \rho^2$ and $r' = r(\alpha - r)$, and via $v = 1/r$ and $v' = -\alpha v + 1$, we find the solution $r(t) = \alpha r_0/(r_0 + (\alpha - r_0)\mathrm{e}^{-\tau})$, and hence

$$\rho(\tau) = \rho_0 \left(\frac{\alpha}{(\alpha - \rho_0^2)\mathrm{e}^{-\alpha\tau} + \rho_0^2}\right)^{1/2}, \tag{2.39}$$

with $\rho(0) = \rho_0 = a/2$, and $\phi(\tau) = \phi_0 = 0$. Figure 2.22 compares some numerical solutions to (2.36) with approximations via (2.37) and illustrates the distortion of the limit cycles of (2.36) from the circles described by (2.37) as ε becomes larger. Approximation results for this special problem can be found in [**Ver96**]. For PDEs on unbounded domains such a perturbation analysis is one of the most powerful tools. In Part IV of this book and such PDEs we prove error estimates for such formal approximations. ⌋

Exercises

Exercises 2.11 and 2.12 should be done with some software for ODE phase portraits, e.g., `xppaut` or `pplane`, and we also recommend to use such software for illustration *after* the analysis for the other planar ODEs, e.g., in Exercises 2.10, 2.15, 2.19, and 2.20.

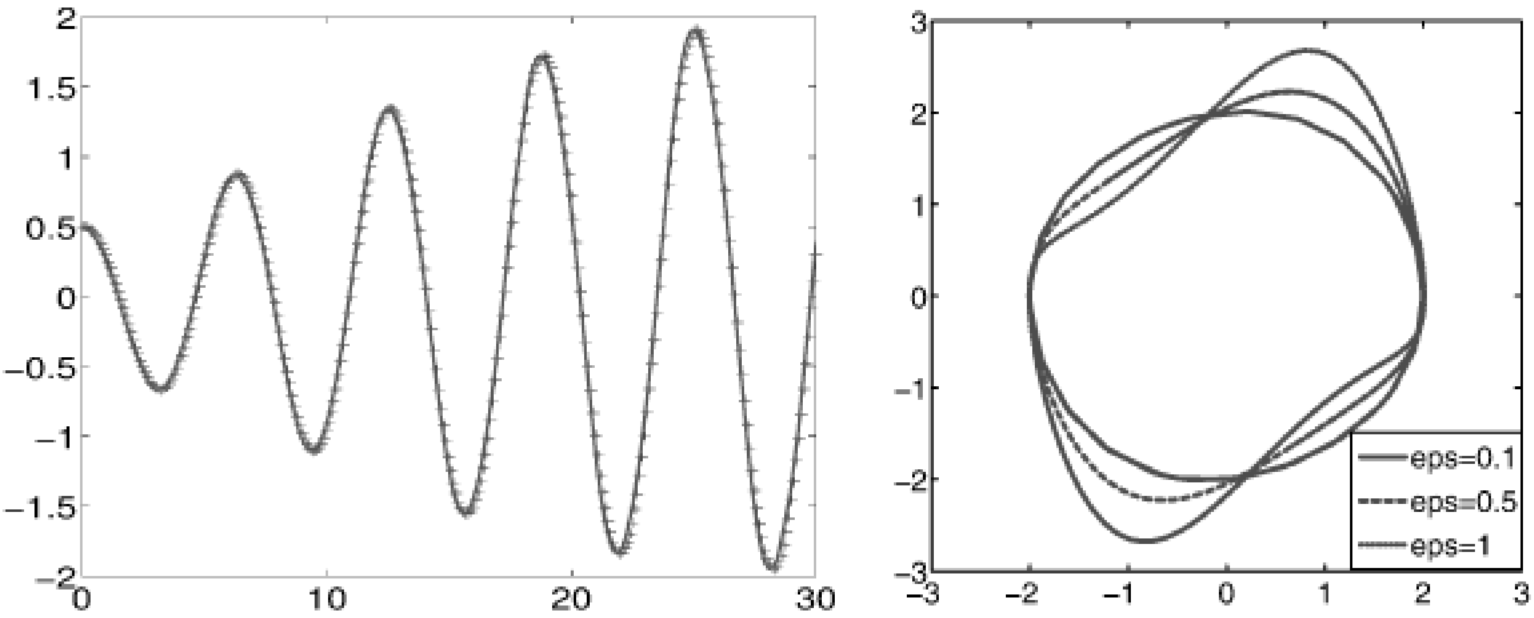


Figure 2.22. Left: numerical solution of (2.36) and approximation via (2.37), $\alpha = 1$, $\varepsilon = 0.2$. Right: Distortion of circle $\rho = 2\sqrt{\alpha}$ by higher-order terms.

2.1. Find the general solution of $\dot{x}(t) = Ax(t)$ with

$$\text{a) } A = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}, \qquad \text{b) } A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \text{c) } A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

2.2. Find the general solutions and the solutions of the initial value problems with $x(0) = \dot{x}(0) = 1$ for: a) $\ddot{x}(t)+3\dot{x}(t)+2x(t) = 0$, b) $\ddot{x}(t)+5\dot{x}(t)+4x(t) = \cos(3t)$.

2.3. Solve the initial value problems

$$\text{a) } \frac{\mathrm{d}}{\mathrm{d}x} y = xy, \quad y(0) = 1, \qquad \text{b) } \frac{\mathrm{d}}{\mathrm{d}x} y = (\cos x)y, \quad y(0) = 1.$$

2.4. Prove that $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$ for $d \times d$-matrices A, B, if $AB = BA$.

2.5. Compute real-valued logarithms of the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} a & a & 1 & 0 \\ -a & a & 0 & 1 \\ 0 & 0 & a & a \\ 0 & 0 & -a & a \end{pmatrix},$$

with $a > 0$. Are the solutions unique?

2.6. Consider $\ddot{y} + 2d\dot{y} + \left(k(t)^2 + d^2\right) y = 0$, with $k(t+T) = k(t)$, $T = r + \frac{\pi}{2}$, and $k(t) = \begin{cases} 0 & \text{for} \quad t \in [0, r), \\ 1 & \text{for} \quad t \in [r, T), \end{cases}$ with $r > 0$. Compute the evolution operator $U(T, 0) = U(T, r)U(r, 0)$. Show that the Floquet-multipliers are given by

$$\rho_{1,2} = \mathrm{e}^{-d(r+\frac{\pi}{2})} \left(-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - 1} \right).$$

Find the domain of stability in which $|\rho_{1,2}| < 1$.

2.7. i) Solve $x_{n+1} = Bx_n$ with $B = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$. ii) Illustrate selected orbits of of $x_{n+1} = Bx_n$ for

$$\text{a) } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{b) } \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \text{c) } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{d) } \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \text{e) } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2.8. Consider the one-dimensional ODEs

a) $\dot{u} = u - u^2$, b) $\dot{u} = -u + 4u^3 - u^5$, c) $\dot{u} = \begin{cases} 0, & \text{if } u = 0, \\ -u^3 \sin(1/u), & \text{if } u \neq 0. \end{cases}$

Find the fixed points and compute their linearization. Which fixed points are stable and which fixed points are unstable? Sketch the phase portraits.

2.9. Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(u) = Au + g(u)$ with

$$Au = (au_1, acu_2, cu_3), \quad g(u) = (0, acu_1u_3, 0),$$

where $a > 1 > c > a^{-1} > 0$. According to the discrete Hartman-Grobman theorem, cf. Remark 2.3.9, there exists a homeomorphism h such that $h^{-1} \circ f \circ h = A$. Show that h cannot be Lipschitz-continuous.
Hint: Clearly $h^{-1} \circ f^n \circ h = A^n$. Show that this implies

$$c^{-n}h_2(u_1, 0, c^n u_3) - a^n h_2(a^{-n}u_1, 0, u_3) = nh_1(u_1, 0, c^n u_3)h_3(a^{-n}u_1, 0, u_3).$$

Next show that $h_2(u_1, 0, 0) = 0$ and $h_2(0, 0, u_3) = 0$, if h is Lipschitz-continuous. Then obtain a contradiction for $n \to \infty$.

2.10. Consider $\dot{x} = y$, $\dot{y} = -cy - x + x^3$, with $c \in \mathbb{R}$ a parameter. Find the fixed points and compute their linearization. Which fixed points are stable and which fixed points are unstable? Sketch the phase portrait for different values of c. Find the stable, the unstable and the center manifolds for the fixed points. ⌋

2.11. Find the possible ω-limit sets for $\dot{x} = y$, $\dot{y} = x + \varepsilon y - x^3 + 0.1x^2y$, with $\varepsilon \in [-0.09, -0.07]$. Compute values of $\varepsilon \in [-0.09, -0.07]$ where a qualitative change of the periodic orbits occurs. *Hint:* Unstable objects can be found by $t \mapsto -t$. ⌋

2.12. Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} (x^2 + \alpha y^2)x \\ (\alpha x^2 + y^2)y \end{pmatrix}.$$

Plot the phase portrait for $\alpha = 1, 5, 10$. Find the fixed points and the periodic solutions. Which of them are stable? Find the maximal $\alpha^* > 1$, such that there exists a non-trivial stable periodic solution for all $\alpha \in [1, \alpha^*]$. (Hint: Consider the phase portrait for $\alpha \in (10, 12)$ by computing the ω-limit set for the initial condition $(x, y) = (0.1, 0.1)$.) Let $\alpha = 12$. Find the fixed points and the associated stable and unstable manifolds. ⌋

2.13. Consider $\dot{u} = f(u)$ with $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Let $\Omega \subset \mathbb{R}^2$ be open and simply connected. Assume the existence of a $b \in C^1(\mathbb{R}^2, \mathbb{R})$ with $\operatorname{div}(bf) > 0$ in Ω. Use the integral law of Gauss to show the non-existence of a periodic orbit in Ω.

2.14. Use the idea from Example 2.4.12 to prove that $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$; has a periodic solution.

2.15. Discuss the stability of the fixed point $(x, y) = (0, 0)$ and sketch the phase portraits for the following systems; compare with $\dot{x} = y, \dot{y} = -x$ and explain the qualitative differences.

a) $\dot{x} = y$, $\dot{y} = -x^3$. b) $\dot{x} = y^{1999}$, $\dot{y} = -x^{1999}$.

Hint for a) Consider $V(x, y) = \alpha x^4 + y^2$ with suitable α.

2.16. Consider $\ddot{x} + \delta(x)\dot{x} + 25x = 0$ with $\delta(x) = 8$ for $|x| > 1$ and $\delta(x) = -6$ for $|x| \leq 1$. In order to show the existence of a periodic orbit consider the Poincaré map $G_1 : S_1 \to S_2$ and $G_2 : S_2 \to S_3$, where $S_1 = \{(x, \dot{x}) : x = -1,\ \dot{x} \geq 0\}$, $S_2 = \{(x, \dot{x}) : x = 1,\ \dot{x} \geq 0\}$ and $S_3 = \{(x, \dot{x}) : x = 1,\ \dot{x} \leq 0\}$. Use then the symmetry of the problem.

2.17. Let Σ' be the set of all $0-1$ sequences $(s_j)_{j\in\mathbb{N}}$ with the following property. If $s_j = 0$, then $s_{j+1} = 1$, i.e., Σ' consists of all sequences without two succeeding zeroes. Prove that:
a) the shift σ maps Σ' into itself;
b) there exists a dense orbit in Σ';
c) the set of periodic orbits is dense in Σ'.

2.18. Prove that the shift σ on Σ_2^+ is conjugated to the tent map $f : [0, 1) \to [0, 1)$ defined by

$$f(x) = \begin{cases} 2x, & \text{for } x \in [0, 1/2), \\ 2x - 1, & \text{for } x \in [1/2, 1). \end{cases}$$

Hint: show that $\phi \circ f = \sigma \circ \phi$ where $\phi(x) = (x_j)_{j\in\mathbb{N}}$ for $x = \sum_{j=1}^{\infty} x_j 2^{-j}$.

2.19. The dynamics of the prey predator system (2.34) is not robust under perturbations. Consider (2.34) with the modification $\dot{x} = x(a(x) - y)$ with $a(x) = a\mathrm{e}^{-\beta x}$ for a $\beta > 0$.
a) Give a biological interpretation of the modification.
b) Show the asymptotic stability of the fixed point $(x, y) = (c, a(c))$.
c) Use the Lyapunov function $\phi(x, y) = x + y - c \ln x - a(c) \ln y$ to prove that all solutions starting with $x(0) > 0$ and $y(0) > 0$ converge towards this fixed point.
d) Sketch the phase portrait.

2.20. Consider the 2-species interaction systems

$$(1)\ \begin{aligned} \dot{u} &= u\left(\tfrac{2}{1+v^2} - u\right), \\ \dot{v} &= v\,(u - v), \end{aligned} \qquad (2)\ \begin{aligned} \dot{u} &= u\left(v - u^2 - 1\right), \\ \dot{v} &= v\,(u - v + 3), \end{aligned} \qquad (3)\ \begin{aligned} \dot{u} &= u - u^2 + \tfrac{u}{1+v}, \\ \dot{v} &= 2v - v^2 + \tfrac{v}{1+u}, \end{aligned}$$

all on $u, v > 0$. For each system, compute the nontrivial fixed point and its linearized stability, and sketch the phase portrait. Classify the systems according to Example 2.6.5.

Chapter 3

Dissipative dynamics

In this chapter we provide the strategy and the tools to tackle dissipative systems, which are characterized by the existence of a compact absorbing set. In such systems very often through so called bifurcations complicated and eventually chaotic dynamics occur if some external parameter is varied. In applications such an external parameter can be for instance an external heating or the concentration of a chemical substance. A typical scenario is as follows. For small values of this parameter all solutions are attracted to some asymptotically stable fixed point. If the value of the parameter is increased the fixed point becomes unstable. Then more complicated dynamics can be expected in a neighborhood of the unstable fixed point, for instance new fixed points or time-periodic solutions may bifurcate, i.e., appear in a neighborhood of the first unstable fixed point. A further increase of the external parameter leads to instabilities of the bifurcating solutions. Then quasi-periodic solutions can occur. The next bifurcation may already lead to chaotic dynamics.

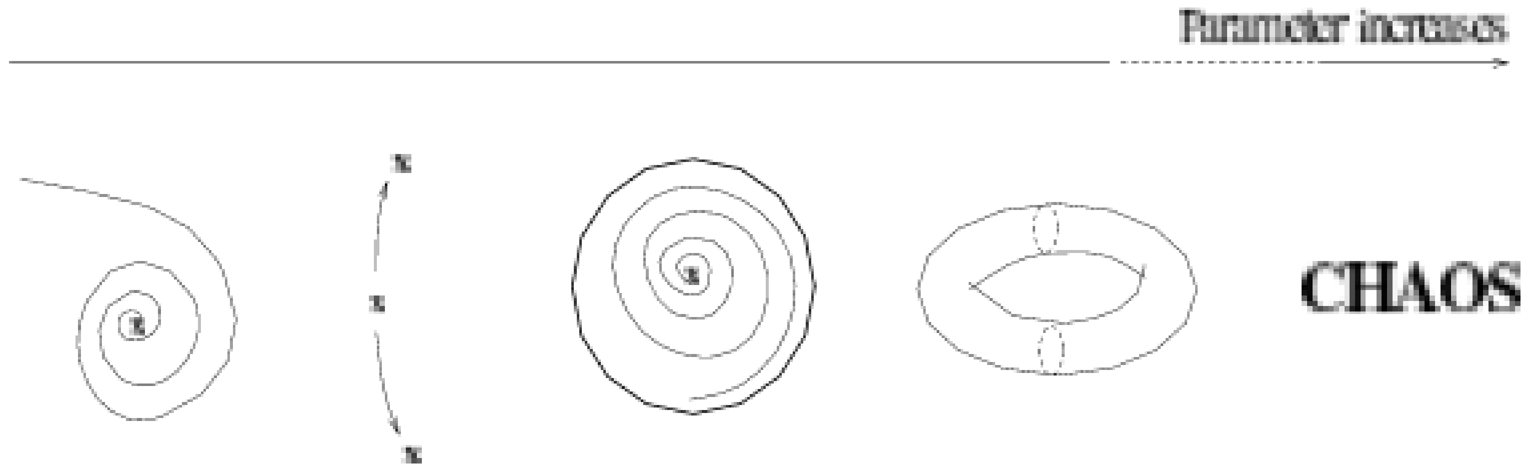


Figure 3.1. Complicated dynamics occurs in dissipative systems as a parameter is increased.

After introducing a number of elementary bifurcations for one- and two-dimensional systems we introduce with the Lyapunov-Schmidt reduction and the center manifold theorem two reduction methods which allow to find these elementary bifurcations in higher dimensional systems, too. Center manifold theory turns out to be a very powerful tool. Besides the construction of the bifurcating solutions it also yields information on their stability. If a fixed point changes from stable to unstable, then all nearby solutions are attracted with some exponential rate towards the center manifold, i.e., the interesting non-trivial dynamics happens on the center manifold of the fixed point. So called normal form transformations allow to analyze the dynamics on the center manifold. We will present this method in the context of the proof of the Hopf bifurcation theorem, i.e., we use it to prove the bifurcation of time-periodic solutions. The chapter is closed by sketching some routes of bifurcations to chaotic behavior in dissipative systems.

3.1. Bifurcations

We present a number of elementary bifurcations and explain how the implicit function theorem and the Lyapunov-Schmidt reduction can be used to prove their occurrence in more complicated systems.

3.1.1. Examples of elementary bifurcations. We start with a globally attracting fixed point which becomes unstable when an external parameter is changed. The following examples are the simplest ones which however turn out to be the 'generic' (see Remark 3.1.10) bifurcations occurring at a fixed point.

Example 3.1.1. (Pitchfork bifurcation of fixed points) Consider

$$\dot{x} = f(x, \mu) = \mu x - x^3,$$

with $x = x(t) \in \mathbb{R}$ and $\mu \in \mathbb{R}$. The linear stability analysis of $x = x_1^* = 0$ gives: $x_1^* = 0$ is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. At $\mu = 0$ a real eigenvalue crosses the imaginary axis and two further fixed points $x_{2,3}^* = \pm\sqrt{\mu}$ bifurcate from $x_1^* = 0$. There is an exchange of stability: for $\mu < 0$, $x_1^* = 0$ is stable; for $\mu > 0$, $x_1^* = 0$ is unstable and $x_{2,3}^* = \pm\sqrt{\mu}$ are stable, since for $\mu > 0$ the linearization $A = (\mu - 3x^2)|_{x=x_{2,3}^*} = -2\mu$ has the negative eigenvalue -2μ. Since the fixed points only exist for $\mu > 0$, where $x_1^* = 0$ is unstable, this bifurcation is called a supercritical pitchfork bifurcation. In case $f(x, \mu) = \mu x + x^3 = 0$ we can explicitly compute the bifurcating unstable branches $x_{2,3}^* = \pm\sqrt{-\mu}$ for $\mu < 0$. These exist where the primary solution $x_1^* = 0$ is stable, and the bifurcation is called a subcritical pitchfork bifurcation. See Figure 3.2. ⌋

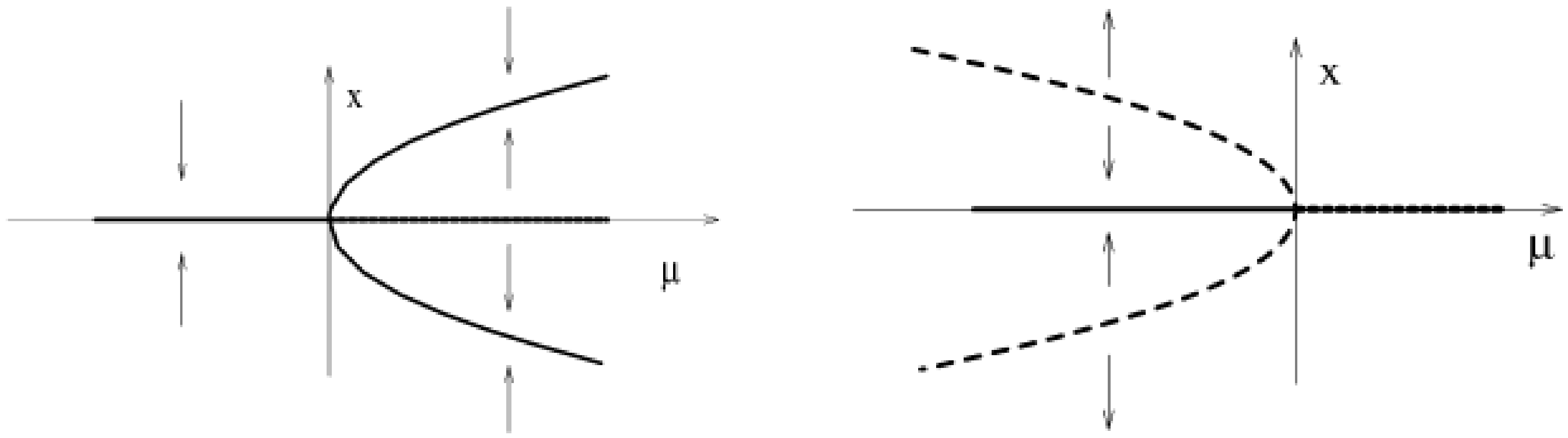


Figure 3.2. Super- and subcritical pitchfork bifurcation of fixed points.

There are two other elementary bifurcations of fixed points, namely the transcritical bifurcation and the saddle-node bifurcation.

Example 3.1.2. (Transcritical bifurcation of fixed points) Consider

$$\dot{x} = \mu x - x^2,$$

with $x = x(t) \in \mathbb{R}$ and $\mu \in \mathbb{R}$. The trivial fixed point $x = x_1^* = 0$ is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. For $\mu = 0$ a real eigenvalue crosses the imaginary axis. There exists another fixed point $x_2^* = \mu$, which coincides with the trivial solution $x_1^* = 0$ for $\mu = 0$. Since in general we know a priori only the trivial solution, we say that the fixed point $x_2^* = \mu$ bifurcates from the trivial solution $x_1^* = 0$. For the transcritical bifurcation an exchange of stability takes place: for $\mu < 0$, $x_1^* = 0$ is stable and $x_2^* = \mu$ is unstable; for $\mu > 0$, $x_1^* = 0$ is unstable and $x_2^* = \mu$ is stable. See Figure 3.3. ⌋

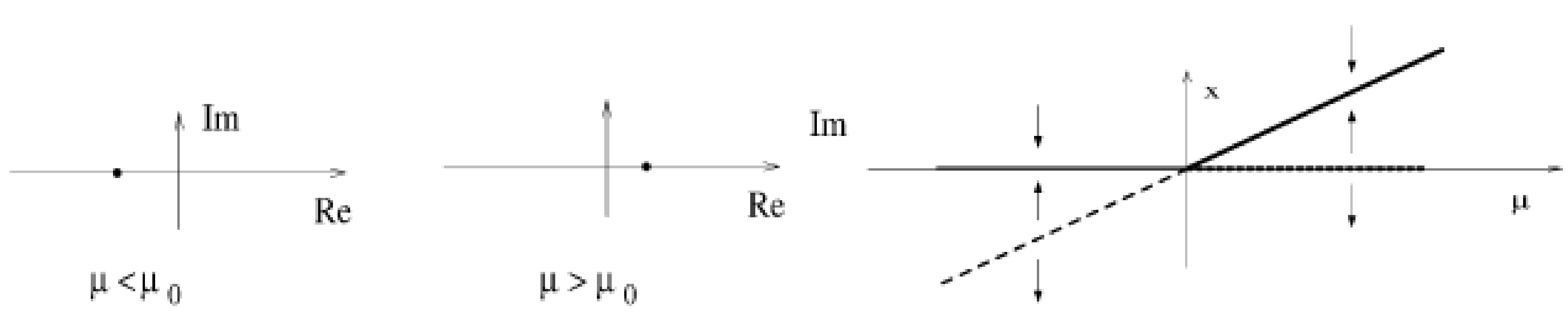


Figure 3.3. A real eigenvalue crosses the imaginary axis leading (here) to a transcritical bifurcation.

Example 3.1.3. (Saddle-node or flip bifurcation of fixed points) Consider

$$\dot{x} = \mu - x^2,$$

with $x = x(t) \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Two fixed points $x_{1,2}^* = \pm\sqrt{\mu}$ appear at $\mu = 0$. The linearization around $x_{1,2}^*$ gives $\mp 2\sqrt{\mu}$. Thus, x_1^* is stable and x_2^* is unstable. See Figure 3.4. The origin of the name saddle-node bifurcation can be seen in $\dot{x} = \mu - x^2$, $\dot{y} = -y$: for this system, $(x_2^*, 0)$ is a saddle and $(x_1^*, 0)$ is a stable node. See Figure 3.4. ⌋

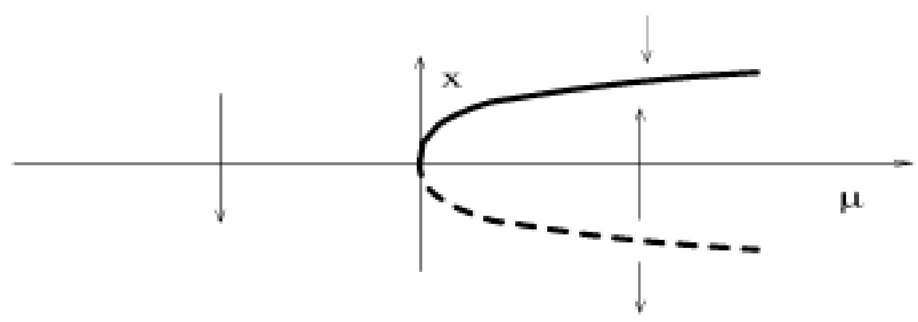


Figure 3.4. Saddle-node-bifurcation.

There is another elementary bifurcation which may occur when a globally attracting fixed point becomes unstable, namely the bifurcation of periodic solutions from a fixed point.

Example 3.1.4. (Hopf bifurcation) Consider

$$\dot{x}_1 = \mu x_1 + x_2 - x_1(x_1^2 + x_2^2) \quad \text{and} \quad \dot{x}_2 = -x_1 + \mu x_2 - x_2(x_1^2 + x_2^2), \tag{3.1}$$

with $x_j(t) \in \mathbb{R}$ and $\mu \in \mathbb{R}$. The linearization $A = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$ around $x = 0$ possesses the eigenvalues $\lambda_{1,2} = \mu \pm \mathrm{i}$, i.e., two complex conjugate eigenvalues cross the imaginary axis for $\mu = 0$. See Figure 3.5. Introducing polar coordinates $x_1 = r\sin\phi$ and $x_2 = r\cos\phi$ with $r \geq 0$ and $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$ gives

$$\dot{r} = \mu r - r^3 \quad \text{and} \quad \dot{\phi} = 1.$$

Hence, a family of periodic solutions

$$\{x = x_{\mathrm{per}}(t, \mu, \phi_0) : x_1 = \sqrt{\mu}\sin(t + \phi_0),\ x_2 = \sqrt{\mu}\cos(t + \phi_0)\}$$

bifurcates from the trivial solution $x = 0$ at $\mu = 0$. This is called a supercritical Hopf bifurcation. For fixed $\mu > 0$ the family attracts every solution with an exponential rate $\mathcal{O}(\exp(-2\mu t))$, see Figure 3.5.

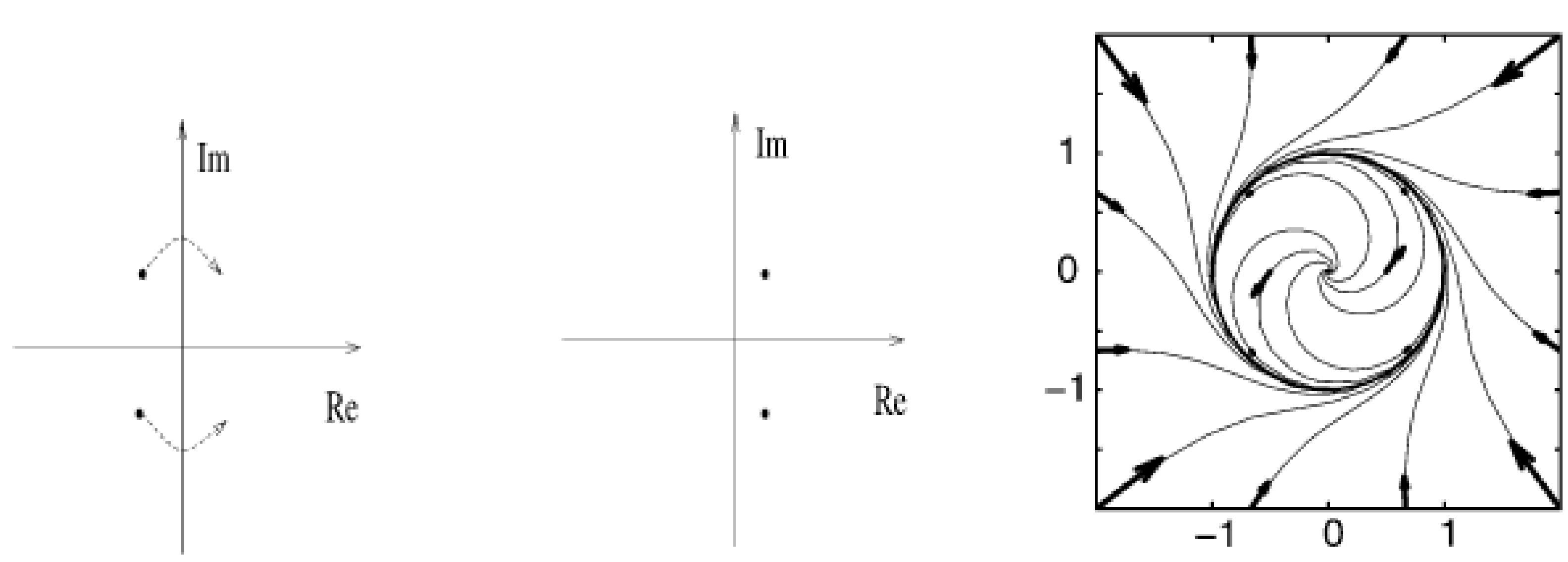


Figure 3.5. Two complex conjugate eigenvalues cross the imaginary axis and the phase portrait for (3.1) for $\mu > 0$.

In §3.3 we shall see that this bifurcation occurs generically when a fixed point loses stability due to two complex conjugate eigenvalues crossing the imaginary axis. ⌋

Example 3.1.5. (Subcritical Hopf bifurcation with turning point) Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (r^2 - r^4) \begin{pmatrix} x \\ y \end{pmatrix},$$

with $r^2 = x^2 + y^2$. The occurrence of $x^2 + y^2$ on the right-hand side suggests the use of polar coordinates which yields $\dot{r} = \varepsilon r + r^3 - r^5$, $\dot{\phi} = 1$. Thus, for $\varepsilon < 0$, with small $|\varepsilon|$, there exists an unstable periodic solution $r \equiv r_0 = \mathcal{O}(\sqrt{|\varepsilon|})$. The second (stable) fixed point $r_1 = \mathcal{O}(1)$ of $\dot{r} = \varepsilon r + r^3 - r^5$ yields a stable periodic solution, see Figure 3.6. The bifurcation is subcritical. The small amplitude non-trivial branch exists in the parameter regime ($\varepsilon < 0$) where the trivial solution is stable. Due to the turning point and the $\mathcal{O}(1)$ amplitude stable periodic orbits in the subcritical regime, this is also called a hard bifurcation since in systems described by such a model the solution may suddenly "jump" to the $\mathcal{O}(1)$ amplitude stable periodic orbit, because in applications noise will push the solution beyond the unstable periodic orbit. In contrast, supercritical bifurcations (stable non-trivial solutions only start to exist after the trivial solution becomes unstable) are soft, since then the bifurcating stable periodic orbits have $\mathcal{O}(\sqrt{\varepsilon})$ amplitude. ⌋

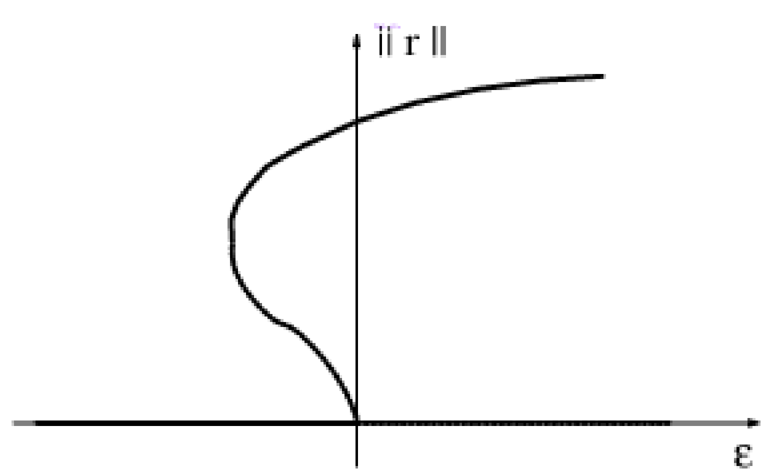


Figure 3.6. Subcritial Hopf bifurcation with turning point

3.1.2. Bifurcations of fixed points. It is the purpose of this section to prove the occurrence of transcritical and pitchfork bifurcations of fixed points from a fixed point in case when the branch of the bifurcating solutions cannot be computed explicitly. The following analysis is based on scaling arguments and the implicit function theorem. The detection of fixed points for the ODE $\dot{x} = f(x, \mu)$, where $\mu \in \mathbb{R}$, leads to the algebraic equation

$$f(x, \mu) = 0. \tag{3.2}$$

Throughout this section we restrict to analytic $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that a solution (x_0, μ_0) of (3.2) is known, i.e., $f(x_0, \mu_0) = 0$, and assume that $\partial_x f(x_0, \mu_0) \neq 0$. Then, by the implicit function theorem there exists a unique smooth solution $x = x(\mu)$ of (3.2), i.e., $f(x(\mu), \mu) = 0$, in a neighborhood of (x_0, μ_0). This solution can be extended outside the neighborhood of (x_0, μ_0) with the same argument until the assumption $\partial_x f(x(\mu), \mu)) \neq 0$ is no longer satisfied. In such a point (x_0, μ_0) a new branch of solutions can bifurcate from this family of solutions $\eta \mapsto (x, \mu)(\eta)$. This so called bifurcation point can be analyzed with the Newton polygon which is explained subsequently. It turns out that generically only two situations for the bifurcations of fixed points can occur, namely the transcritical bifurcation from Example 3.1.2 or the pitchfork bifurcation from Example 3.1.1.

Scaling arguments. One way to establish the existence of bifurcating solutions in the general case are scaling arguments and the implicit function theorem.

Example 3.1.6. Let $f(x, \mu) = \mu x + x^2 + \sin x$. Then $f(0, \mu) = 0$ for all $\mu \in \mathbb{R}$. Hence, $x = x_1^* = 0$ is the trivial solution for all $\mu \in \mathbb{R}$. For all values of $\mu \in \mathbb{R}$ we have $\partial_\mu f|_{x=0} = 0$. Hence, it is sufficient to consider

$$\partial_x f|_{x=0} = (\mu - 2x + \cos x)|_{x=0} = \mu + 1.$$

Thus, a bifurcation can only take place when $\mu + 1 = 0$. Therefore, we introduce the small bifurcation parameter $\alpha = \mu + 1$. In order to find Example 3.1.2 in the present problem we rescale $x = \alpha y$ and introduce

$$F(y, \alpha) = \alpha^{-2} f(\alpha y, 1 + \alpha) = y + y^2 + \mathcal{O}(\alpha).$$

Thus, for $\alpha = 0$ we have the simple equation $F(y, 0) = y + y^2$ having the solutions $y_1^* = 0$ and $y_2^* = -1$. According to

$$\partial_y F|_{(y,\alpha)=(y_j^*,0)} = (1 + 2y)|_{(y,\alpha)=(y_j^*,0)} \neq 0$$

we can apply the implicit function theorem to solve $F = 0$ in a neighborhood of $(y, \alpha) = (y_j^*, 0)$ for y and and obtain $y_1^* = 0 + \mathcal{O}(\alpha)$ and $y_2^* = -1 + \mathcal{O}(\alpha)$. Hence, beside the trivial solution $x_1^* = 0$ we also found the bifurcating solution $x_2^* = \alpha + \mathcal{O}(\alpha^2)$. ⌋

Example 3.1.7. Consider $f(x, \mu) = \mu x + \sin x$. Again $x = x_1^* = 0$ becomes unstable at $\mu = -1$. Let $\alpha^2 = \mu + 1$ and $x = \alpha y$. The rescaled problem

$$F(y, \alpha) = \alpha^{-3} f(\alpha y, 1 + \alpha^2) = y - \frac{1}{6} y^3 + \mathcal{O}(\alpha^2) = 0$$

can be explicitly solved for $\alpha = 0$. Using the implicit function theorem we obtain $y_1^* = 0 + \mathcal{O}(\alpha^2)$ and $y_{2,3}^* = \pm\sqrt{6} + \mathcal{O}(\alpha^2)$, hence $x_1^* = 0$ and $x_{2,3}^* = \pm\sqrt{6}\alpha + \mathcal{O}(\alpha^3)$ for $\alpha > 0$. ⌋

Also more general situations can be handled by scaling arguments.

Example 3.1.8. Consider

$$f(x, \varepsilon) = x^2 + x\varepsilon + \varepsilon^3 = 0. \tag{3.3}$$

We have the trivial solution $(0, 0)$ for which the assumptions of the implicit function theorem are not satisfied, i.e., $\partial_x f(0,0) = \partial_\varepsilon f(0,0) = 0$. Again we are interested in non-trivial solutions $x = x(\varepsilon)$ near the origin.

We make the ansatz $x(\varepsilon) = \varepsilon v(\varepsilon)$ and obtain

$$F(v, \varepsilon) = \varepsilon^{-2} f(\varepsilon v, \varepsilon) = v^2 + v + \varepsilon = 0.$$

For $\varepsilon = 0$ we find the non-trivial solution $v_1^* = -1$. We additionally have $\partial_v F(-1, 0) = -1 \neq 0$ such that we can apply the implicit function theorem and obtain a smooth solution $v = v_1^*(\varepsilon) = -1 + \mathcal{O}(\varepsilon)$. Hence, we find a non-trivial solution $x_1^* = -\varepsilon + \mathcal{O}(\varepsilon^2)$ for $f = 0$. However, the ansatz $x(\varepsilon) = \varepsilon^2 v(\varepsilon)$ yields

$$F(v, \varepsilon) = \varepsilon^{-3} f(\varepsilon^2 v, \varepsilon) = \mathcal{O}(\varepsilon) + v + 1 = 0.$$

For $\varepsilon = 0$, we find the non-trivial solution $v_2^* = -1$ and $\partial_v F(-1, 0) = 1 \neq 0$. Hence, we can apply the implicit function theorem and obtain a smooth solution $v = v_2^*(\varepsilon) = -1 + \mathcal{O}(\varepsilon)$. Therefore, we found a second curve of non-trivial solutions $x_2^* = -\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ for $f = 0$. The expansions correspond to the solutions

$$x_{1,2}(\varepsilon) = -\frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} - \varepsilon^3}$$

of (3.3), which only can be computed explicitly since (3.3) is a second order polynomial w.r.t. x. ⌋

The Newton-polygon. In the last example there exist at least two curves of non-trivial solutions. Since we have a polynomial in the example we can be sure that we found all solutions. For non-polynomial problems the scaling argument can be made rigorous with the help of the Weierstrass preparation theorem which allows to bring analytic f into a polynomial form w.r.t. one of the variables, cf. [**CH82**, §2.8]. With this preparation it is then clear that the solutions which we will find with the scaling arguments are the only non-trivial solutions near $(x, \varepsilon) = (0, 0)$.

A systematic approach to find the scalings is as follows. Assume that f can be expanded in some convergent power series near the origin, i.e., $f(x, \varepsilon) = \sum_{m,n}^{\infty} a_{mn} x^m \varepsilon^n$. Whenever the coefficient $a_{mn} \in \mathbb{R}$ is nonzero, make a dot at $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then take the lower convex hull of all dots in the $\mathbb{N}_0 \times \mathbb{N}_0$-plane. This hull is the so called Newton-polygon with finitely many line segments with endpoints (m_i, n_i) and (m_{i+1}, n_{i+1}) and slopes $-\alpha_i$. Associated with each of these lines there are p_i solutions $x_i^*(\varepsilon) = \varepsilon^{\alpha_i} v_i^*(\varepsilon)$ of $f(x, \varepsilon) = 0$, where $p_i = m_i - m_{i-1}$.

Example 3.1.9. The Newton polygon for

$$f(x, \varepsilon) = x^3 + 3x^2\varepsilon + 2x\varepsilon^2 + \varepsilon^5 = 0$$

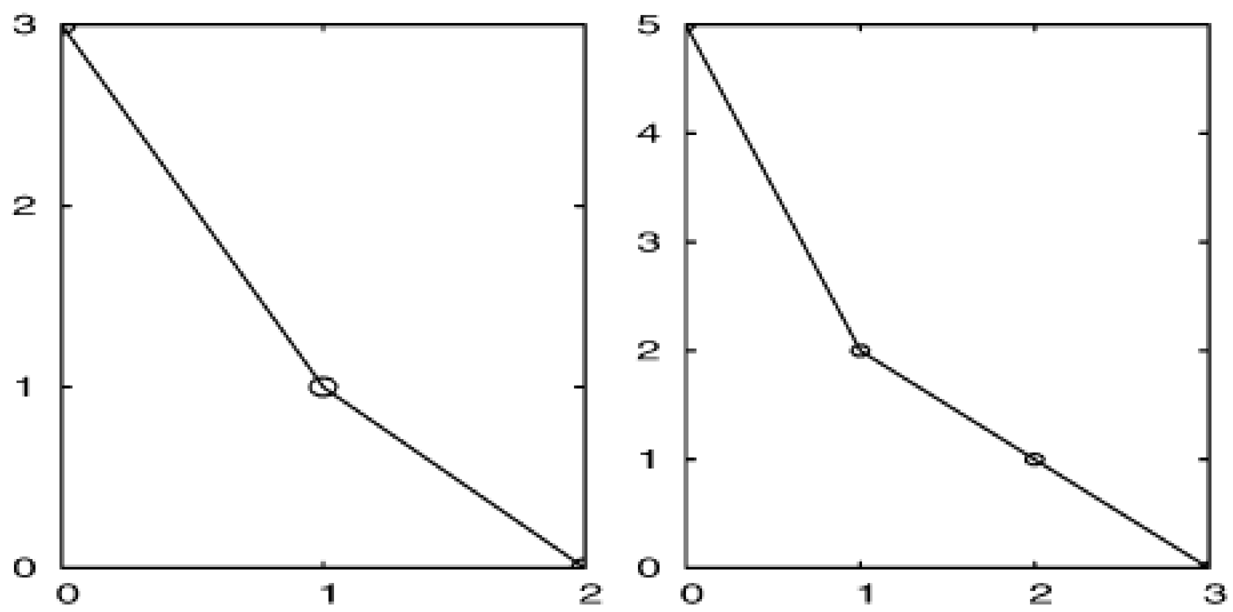


Figure 3.7. Newton polygons for $f(x,\varepsilon) = x^2+x\varepsilon+\varepsilon^3$ and for $f(x,\varepsilon) = x^3 + 3x^2\varepsilon + 2x\varepsilon^2 + \varepsilon^5$.

yields $\alpha_1 = 3$, $p_1 = 1$ and $\alpha_2 = 1$, $p_2 = 2$. With the ansatz $x(\varepsilon) = \varepsilon^\alpha v(\varepsilon)$ we obtain

$$\varepsilon^{3\alpha}v^3 + 3\varepsilon^{1+2\alpha}v^2 + 2\varepsilon^{2+\alpha}v + \varepsilon^5 = 0.$$

The first three terms are of the same leading order for $\alpha = 1$. The third and the fourth term are of the same leading order for $\alpha = 3$.

For $\alpha = 1$ we obtain $F(v,\varepsilon) = v^3 + 3v^2 + 2v + \mathcal{O}(\varepsilon^2) = 0$. For $\varepsilon = 0$ we find the non-trivial solutions $v_1^* = -1$ and $v_2^* = -2$. Since $\partial_v F(-1,0) = (3v^2+6v+2)|_{v=-1} = -1 \neq 0$ and $\partial_v F(-2,0) = 2 \neq 0$ by the implicit function theorem we find the non-trivial solutions $v_1^*(\varepsilon) = -1 + \mathcal{O}(\varepsilon^2)$ and $v_2^*(\varepsilon) = -2 + \mathcal{O}(\varepsilon^2)$ or equivalently $x_1^*(\varepsilon) = -\varepsilon + \mathcal{O}(\varepsilon^3)$ and $x_2^*(\varepsilon) = -2\varepsilon + \mathcal{O}(\varepsilon^3)$.

For $\alpha = 3$, we obtain $F(v,\varepsilon) = \mathcal{O}(\varepsilon^2)+2v+1 = 0$. For $\varepsilon = 0$, we find the non-trivial solution $v_3^* = -1/2$. Since $\partial_v F(-1/2,0) = 2 \neq 0$ by the implicit function theorem we find the non-trivial solution $v_3^*(\varepsilon)=-1/2 + \mathcal{O}(\varepsilon^2)$ or equivalently $x_3^*(\varepsilon)=-\varepsilon^3/2 + \mathcal{O}(\varepsilon^5)$. ⌋

Remark 3.1.10. (The genericity of transcritical and pitchfork bifurcations) If w.l.o.g. we assume that the trivial solution is given by $x = x^* = 0$, then there exists a smooth function $g : \mathbb{R}^2 \to \mathbb{R}$ with $f(x,\mu) = xg(x,\mu)$. If we further assume that the bifurcation point is given by $(x,\mu) = (0,0)$, then $\partial_x f(0,0) = \partial_\mu f(0,0) = 0$. This gives the condition $g(0,0) = 0$ such that

$$g(x,\mu) = g_{10}x + g_{01}\mu + g_{20}x^2 + g_{11}x\mu + g_{02}\mu^2 + \mathcal{O}(|x|^3 + |\mu|^3),$$

with coefficients $g_{ij} \in \mathbb{R}$. Generically we have $g_{10} \neq 0$ and $g_{01} \neq 0$ such that we find a bifurcating branch with $x = -g_{10}^{-1}g_{01}\mu+\mathcal{O}(\mu^2)$, i.e., a transcritical bifurcation. However, symmetries such as $f(x,\mu) = -f(-x,\mu)$ can force $g_{10} = 0$. Solving the equation $g(x,\mu) = 0$ w.r.t. μ and using the Newton polygon we find by the ansatz $\mu(x) = x^2 s(x)$ that $g_{01}\mu$ and $g_{20}x^2$ are of the same order. We obtain

$$G(x,s) = x^{-2}g(x,x^2 s) = g_{01}s + g_{20} + \mathcal{O}(x) = 0$$

and hence $s = -g_{01}^{-1}g_{20}$ such that $\mu = -g_{01}^{-1}g_{20}x^2 + \mathcal{O}(x^3)$, i.e., depending on the sign of $g_{01}^{-1}g_{20}$ a sub- or a supercritical pitchfork bifurcation occurs. More coefficients can vanish, but this is a degenerated situation, which requires more symmetries.

By a small perturbation the vanishing coefficients can be made non-zero. This is called unfolding of the bifurcation, cf. [**GS85**, Chapter III]. Such unfoldings are robust w.r.t. other small perturbations, i.e., additional parameters different from the unfolding parameters will not change the solution set qualitatively. ⌋

3.1.3. The Lyapunov-Schmidt reduction. We consider now an ODE

$$\dot{x} = f(x, \mu)$$

with $x(t) \in \mathbb{R}^d$ in case $d > 1$ under the assumption that at $\mu = \mu_0$ one simple eigenvalue crosses the imaginary axis and that all other eigenvalues have negative real part. See Figure 3.8. We remark that for the subsequent analysis it is sufficient that all eigenvalues except of one are bounded away from the imaginary axis, i.e., eigenvalues with positive real part are allowed, too.

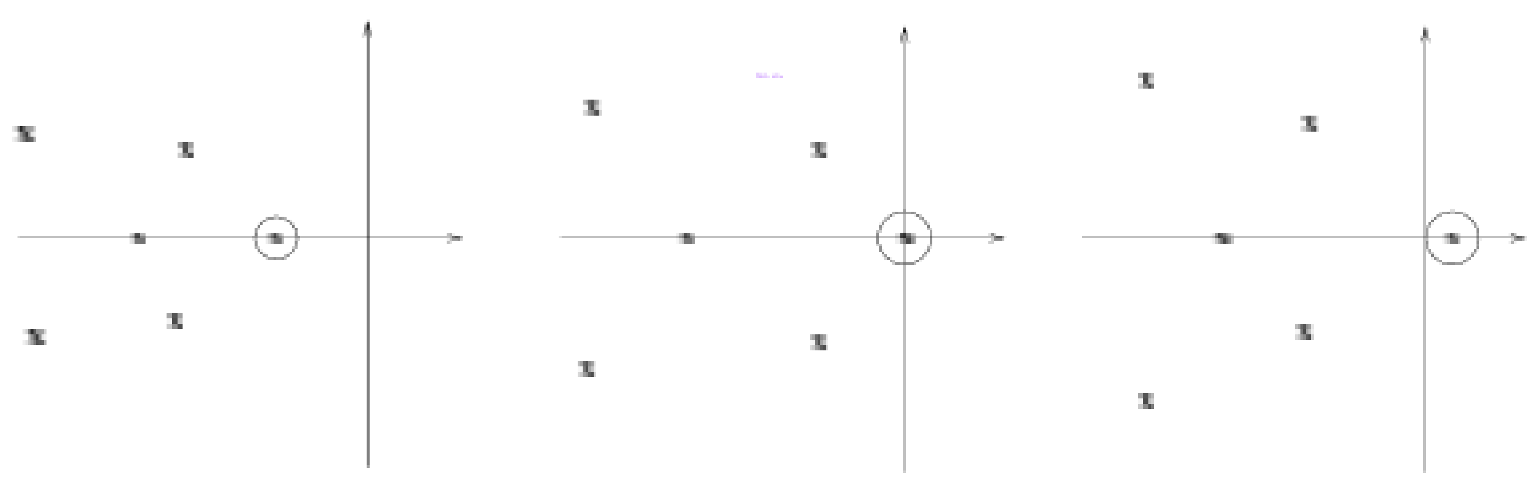


Figure 3.8. Spectral situation in case of a fixed point becoming unstable via a pitchfork or transcritical bifurcation.

In order to establish a pitchfork or a transcritical bifurcation we use the so called Lyapunov-Schmidt method, which allows to reduce the d-dimensional problem $f(x, \mu) = 0$ to a one-dimensional one. So far we have restricted ourselves to problems $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In principle, the assertions from §3.1.2 remain valid also in the general case $f : B \times \mathbb{R} \to B$ with B a Banach space. This means that under the previous spectral assumption generically only transcritical and pitchfork bifurcations occur. In contrast to the examples above, x and μ are no longer equivalent. We distinguish between the variable x and the parameter μ.

Only to avoid a number of technicalities we restrict ourselves in the following to $B = \mathbb{R}^d$. By the implicit function theorem we can compute a solution $x = x(\mu)$ for growing μ until $M = \partial_x f(x(\mu), \mu) \in \mathbb{R}^{d\times d}$ is no longer invertible. We denote this point by (x_0^*, μ_0). Under the previous spectral assumption non-invertibility is equivalent to the fact that exactly one of the d eigenvalues of M is zero. W.l.o.g. we assume that the associated eigenvector is given by e_1. In order to apply the implicit function theorem in this situation, we split the system into two parts, namely into a part where the implicit function theorem can be applied and into a part where it cannot be applied. Denote by P_1 a projection on $\operatorname{span}\{\mathrm{e}_1\} = (1, 0, \ldots, 0)$ and let $P_2 = I - P_1$. Moreover, we set $x_j = P_j x$, denote by Q_2 a projection on the range of M, and let $Q_1 = I - Q_2$. Then, we consider

$$Q_1 f((x_1, x_2), \mu) = 0, \quad \text{and} \quad Q_2 f((x_1, x_2), \mu) = 0.$$

We find that $\partial_{x_2} Q_2 f(x_0, \mu_0) \in \mathbb{R}^{d-1 \times d-1}$ is invertible since it possesses the $(d-1)$ non-zero eigenvalues of M. Hence, the second equation can be solved locally w.r.t. x_2, i.e., there exists a solution $x_2 = x_2(x_1, \mu)$. Inserting this solution into the first equation gives the so called reduced bifurcation problem

$$\widetilde{f}(x_1, \mu) = Q_1 f((x_1, x_2(x_1, \mu)), \mu) = 0.$$

After this so called Lyapunov-Schmidt reduction we are in the same situation as in the previous section. We have to find the zeroes of a smooth function $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}$. If symmetries are present in the problem, then the projections can be chosen in such a way that the symmetries are preserved by the reduction [**GS85**, Chapter VII.3].

Example 3.1.11. Consider

$$\begin{aligned} f_1(x, y, \varepsilon) =& \varepsilon x - yx - x^3 = 0, \\ f_2(x, y, \varepsilon) =& y + 2x^2 + y^2 \varepsilon^2 = 0. \end{aligned}$$

The origin $(x, y) = (0, 0)$ is a solution for all $\varepsilon \in \mathbb{R}$, and we are interested in non-trivial solutions close to it. The linearization

$$\partial_{(x,y)}(f_1, f_2)|_{(x,y)=(0,0)} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$$

has the eigenvalues ε and 1. Hence, a bifurcation is only possible for $\varepsilon = 0$. The kernel is given by $\{(x, 0) : x \in \mathbb{R}\}$ and the range by $\{(0, y) : y \in \mathbb{R}\}$. Thus, the above system is already in the form needed for the Lyapunov-Schmidt reduction, and the second equation can be solved w.r.t. y. In order to obtain an approximate solution we consider an iteration of the second equation, namely

$$y = -2x^2 - y^2\varepsilon^2 = -2x^2 - \left(-2x^2 - y^2\varepsilon^2\right)^2 \varepsilon^2 = -2x^2 + \mathcal{O}\left(\varepsilon^2 x^4\right).$$

Inserting the solution $y = -2x^2 + \mathcal{O}(\varepsilon^2 x^4)$ into the first equation gives the bifurcation equation

$$f_1(x, y(x, \varepsilon), \varepsilon) = \varepsilon x - \left(-2x^2 + \mathcal{O}\left(\varepsilon^2 x^4\right)\right) x - x^3 = \varepsilon x + x^3 + \mathcal{O}\left(\varepsilon^2 x^5\right) = 0.$$

Dividing this equation by x gives $\varepsilon + x^2 + \mathcal{O}\left(\varepsilon^2 x^4\right) = 0$ which can be analyzed by the Newton polygon. We find a subcritical pitchfork bifurcation, i.e., non-trivial solutions $x^*(\varepsilon) = \pm\sqrt{-\varepsilon} + \mathcal{O}(\varepsilon)$ and $y^*(\varepsilon) = \mathcal{O}(\varepsilon)$ for $\varepsilon < 0$. ⌋

Consequence. When a fixed point becomes unstable by a simple eigenvalue crossing the imaginary axis, generically a transcritical or a pitchfork bifurcation occurs. Hence, even in higher-dimensional phase spaces for this spectral situation no new bifurcations can occur.

Remark 3.1.12. The Lyapunov-Schmidt reduction has certain disadvantages. It does not provide information about the stability of bifurcating solutions. Treating Hopf bifurcations via Lyapunov-Schmidt reduction leads to an infinite-dimensional problem. In order to find $2\pi/\omega$-time-periodic solutions of the ODE $\dot{x}(t) = f(x, \mu)$ the problem is transferred by using Fourier series $x(t) = \sum_{j\in\mathbb{Z}} \widehat{x}_j e^{i\omega j t}$ to an infinite-dimensional stationary problem for the Fourier coefficients $\widehat{x}_j$. With the help of the Lyapunov-Schmidt reduction the problem can be reduced to a two-dimensional one, cf. [**CH82**, §1.4]. The construction of homoclinic and heteroclinic solutions with this method again leads to an infinite-dimensional problem and requires special properties of the underlying ODE, cf. [**PS07**]. ⌋

3.2. Center manifold theory

Center manifold theory is an alternative way to find the elementary bifurcations from above also in higher space dimensions. Additionally, it often yields information on the stability of the bifurcating solutions. Moreover, in contrast to the Lyapunov-Schmidt reduction with this method Hopf bifurcations and the occurrence of small amplitude homoclinic and heteroclinic solutions can be handled as finite-dimensional problems.

If a fixed point becomes unstable, all solutions are attracted with some exponential rate towards the center manifold, i.e., the interesting non-trivial dynamics happens on the center manifold of the unstable fixed point. In general, only polynomial approximations of the vector field on the center manifold are known. If the center manifold has two and more space dimensions, so called normal form transformations help us to analyze the dynamics on the center manifold. In the next section we use center manifold theory and normal forms to prove a general Hopf bifurcation theorem.

We already formulated in Theorem 2.3.19 the invariant manifold theorem. The part about the center manifold is now made more precise. We consider

$$\begin{aligned} \dot{u}_c &= B_c u_c + \widetilde{g}_c(u_c, u_s, u_u), \\ \dot{u}_s &= B_s u_s + \widetilde{g}_s(u_c, u_s, u_u), \\ \dot{u}_u &= B_u u_u + \widetilde{g}_u(u_c, u_s, u_u), \end{aligned} \tag{3.4}$$

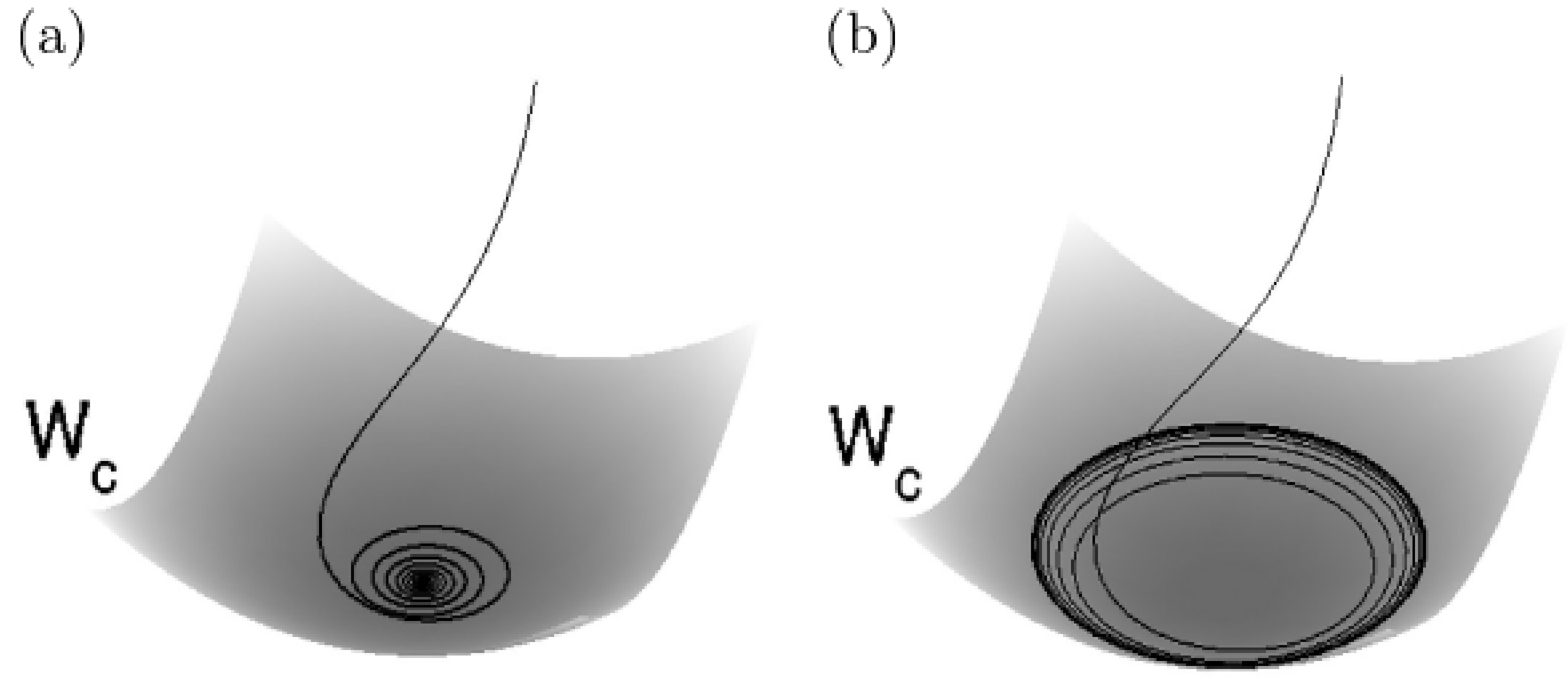


Figure 3.9. When a fixed point becomes unstable the bifurcating solutions can be found on the exponentially attracting center manifold. (a),(b) before and after the bifurcation of a stable periodic orbit on M_c.

with $u_c \in E_c = \mathbb{R}^{d_c}, u_s \in E_s = \mathbb{R}^{d_s}, u_u \in E_u = \mathbb{R}^{d_u}$ some finite-dimensional vectors, B_c a matrix with eigenvalues on the imaginary axis, B_s a matrix with eigenvalues with negative real part, B_u a matrix with eigenvalues with positive real part, and $\widetilde{g}_c : \mathbb{R}^d \to E_c, \widetilde{g}_s : \mathbb{R}^d \to E_s$ and $\widetilde{g}_u : \mathbb{R}^d \to E_u$, $d = d_c + d_s + d_u$, are C^{r+1}-maps without constant and linear terms.

Theorem 3.2.1. (Center manifold theorem) *There exists a neighborhood $U \subset E_c$ of $u_c = 0$ and a C^r-map $h : U \ni u_c \mapsto h(u_c)$ such that the manifold*

$$W_c = \{u = u_c \oplus h(u_c) : u_c \in U,\ (u_s, u_u) = h(u_c)\}$$

is invariant under the flow of (3.4). *W_c is called the center manifold. The reduced flow is determined by*

$$\dot{u}_c = B_c u_c + \widetilde{g}_c(u_c, h_s(u_c), h_u(u_c)). \tag{3.5}$$

The function h contains no constant and no linear terms w.r.t. u_c such that the center manifold W_c is tangential to the central subspace E_c associated to the eigenvalues with vanishing real part. In general the center manifold is not unique.

Some parts of the theorem, namely the invariance, the existence, and Lipschitz-continuity instead of r-times differentiability of the center manifold will be proven in §13.1. Here we will discuss some of the assertions of the theorem and concentrate on its application by giving a number of examples. With the first example we explain how center manifold theory can be used to handle bifurcation problems although the central eigenvalues are only on the imaginary axis for one particular value of the bifurcation parameter

Example 3.2.2. For μ close to zero consider the trivial decoupled system

$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y. \tag{3.6}$$

For $\mu < 0$, the origin $(x, y) = (0, 0)$ is stable. For $\mu = 0$, we find the one-dimensional center manifold $W_c = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. In order to handle non-zero values of μ with the center manifold theorem the above system is extended to

$$\dot{x} = \mu x - x^3, \quad \dot{y} = -y, \quad \dot{\mu} = 0.$$

For this extended system we find the two-dimensional center manifold $W_c = \{(\mu, x, y) \in \mathbb{R}^3 : y = 0\}$. Note that after introducing $\dot{\mu} = 0$ the term μx is no longer a linear, but a nonlinear term. Since $\dot{\mu} = 0$ implies that μ is a constant, the two-dimensional center manifold is foliated by one-dimensional invariant manifolds. See Figure 3.10. Hence, the additional equation $\dot{\mu} = 0$ can be canceled again and on the two-dimensional center manifold μ can be considered again as a parameter. Therefore, by applying the center manifold theorem in a sloppy way the two-dimensional bifurcation problem (3.6) can be reduced to

$$\dot{x} = \mu x - x^3$$

in the one-dimensional center manifold $W_c = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. Obviously, the reduction is trivial in this case, i.e., $h = 0$. In summary, bifurcation problems can be handled with the help of the center manifold theorem by introducing the equation $\dot{\mu} = 0$. ⌋

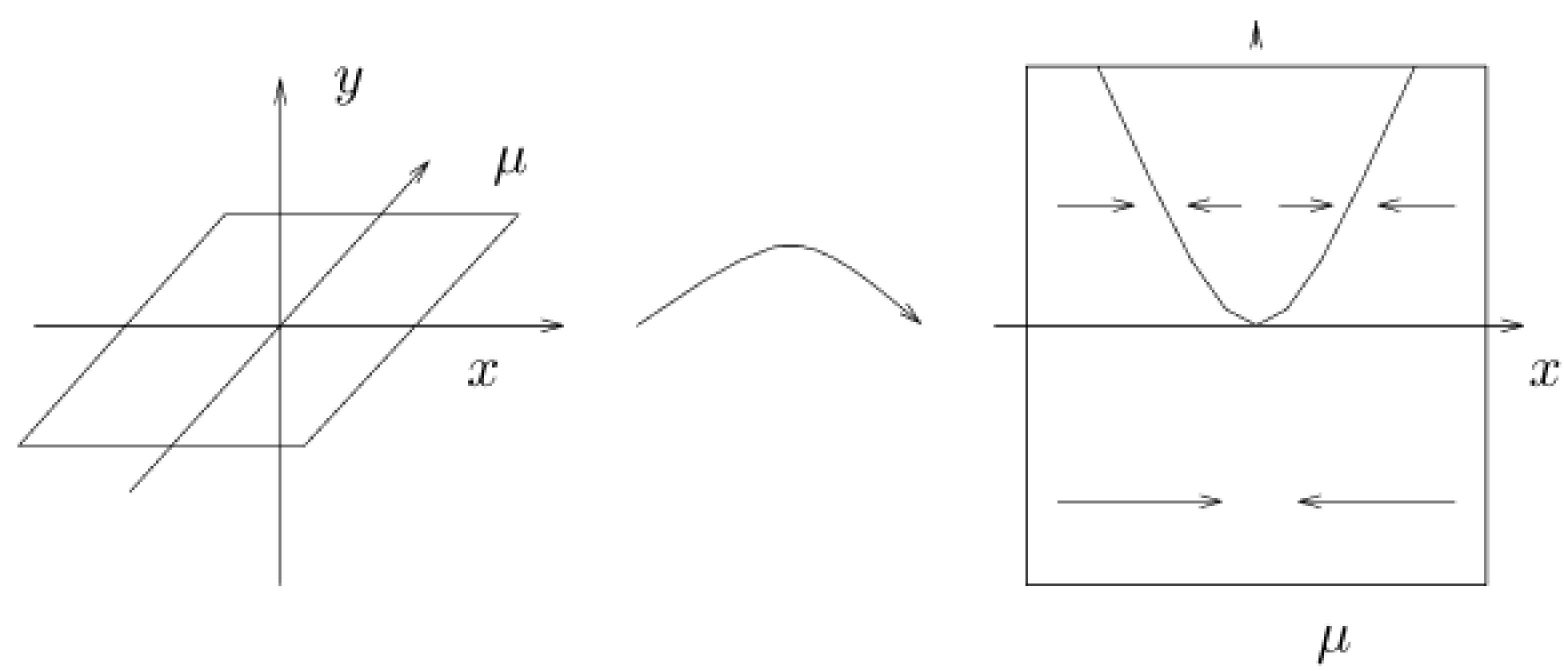


Figure 3.10. Reduction of the system $\dot{x} = \mu x - x^3$, $\dot{y} = -y$, $\dot{\mu} = 0$ to a two-dimensional center manifold which is foliated by invariant one-dimensional manifolds.

The next example shows how to compute approximations of the reduction function h and of the reduced system on the center manifold.

Example 3.2.3. For μ close to zero consider

$$\dot{x} = \mu x + x^3 - xy, \qquad \dot{y} = -y + 2x^2.$$

Like above we extend the system by the equation $\dot{\mu} = 0$. The linearized system is given by

$$\dot{x} = 0, \quad \dot{y} = -y, \quad \dot{\mu} = 0,$$

and hence $E_c = \{(\mu, x, y) \in \mathbb{R}^3 : y = 0\}$. Therefore, we make the ansatz

$$y = h(x, \mu) = ax^2 + b\mu x + c\mu^2 + \mathcal{O}(|\mu|^3 + |x|^3),$$

and from $\dot{y} = -y + 2x^2$ we obtain $2ax\dot{x} + \mu\dot{x} + \ldots = -(ax^2 + b\mu x + c\mu^2 + \ldots) + 2x^2$. Since $\dot{x} = \mu x + \ldots$, by comparing the coefficients it follows

$$x^2 : \ 0 = -a + 2, \quad x\mu : \ 0 = -b, \quad \mu^2 : \ 0 = -c, \quad \ldots$$

As a general rule, no powers μ^n without x can occur in h, and therefore

$$W_c = \{(\mu, x, y) \in \mathbb{R}^3 : y = 2x^2 + \mathcal{O}(|\mu|x^2 + |x|^3)\}.$$

Moreover, the function h cannot contribute to the quadratic terms of the reduced system which here is given by

$$\dot{x} = \mu x + x^3 - x(2x^2) + \text{h.o.t} = \mu x - x^3 + \text{h.o.t.},$$

i.e., the fixed point $(x, y) = (0, 0)$ is stable also for $\mu = 0$. We explain below that stability on the center manifold implies stability in the full system in such a situation. At $\mu = 0$ a supercritical pitchfork bifurcation occurs. ⌋

The following two examples are about the non-uniqueness and the non-smoothness of center manifolds.

Example 3.2.4. In order to illustrate the non-uniqueness of the center manifold we consider

$$\dot{x} = x^2, \qquad \dot{y} = -y. \tag{3.7}$$

Obviously, the central subspace E_c is given by the x-axis. The solutions of the ODEs are given by $x(t) = \frac{x_0}{1-tx_0}$ and $y(t) = y_0 \mathrm{e}^{-t}$. Elimination of time t yields $y(x) = (y_0 \mathrm{e}^{-1/x_0})\mathrm{e}^{1/x}$. For $x < 0$ we have $\lim_{x\to 0, x<0} y^{(n)}(x) = 0$, i.e., every solution approaches the origin in a flat way. For $x > 0$ we find that $y = 0$ is the only solution which approaches the origin. Thus, we find infinitely many different C^∞-center manifolds which are tangential to E_c at the origin by glueing together the obits in the left half plane with the positive real axis. This shows that center manifolds are non-unique in general. The only analytic center manifold, i.e., with a convergent power series, is the x-axis. See Figure 3.11 for the phase portrait of (3.7). ⌋

Example 3.2.5. In order to illustrate the non-smoothness (and non-uniqueness) of the center manifold we consider

$$\dot{x} = -\mu x, \quad \dot{y} = -y, \quad \dot{\mu} = 0$$

with $0 < \mu < 1$. The vector field is C^∞ and E_c is given by the (x, μ)-plane. Obviously $W_c = E_c$ is a center manifold. However, the trajectories satisfy $\frac{\mathrm{d}}{\mathrm{d}x} y = \mu \frac{y}{x}$ and are given through $y(x) = C|x|^{1/\mu}$. If $r < 1/\mu < r + 1$ with $r \in \mathbb{N}$, then the trajectories for fixed μ are in C^r, but not in C^{r+1}. Each of

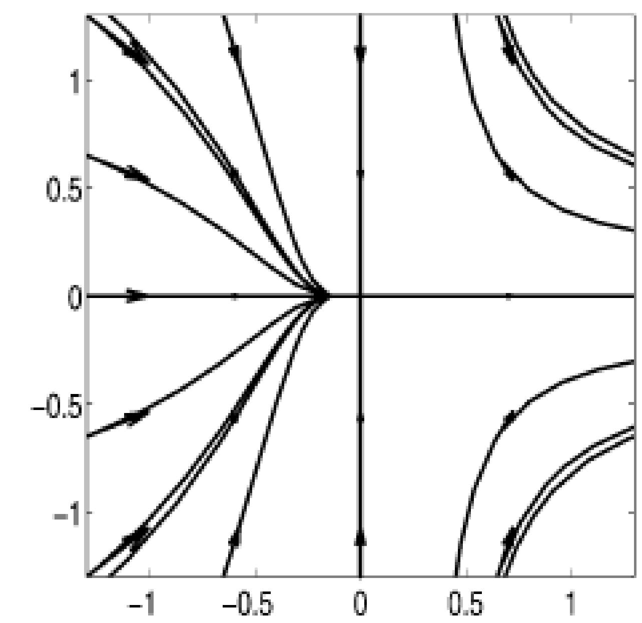


Figure 3.11. The phase portrait of $\dot{x} = x^2, \quad \dot{y} = -y$.

these curves is tangent to $y = 0$ and so the whole of these trajectories forms another center manifold. In a ball of radius $\mu < 1/r$ this center manifold is C^r, i.e., the larger r is chosen, the smaller is the center manifold. ⌋

Remark 3.2.6. In case of no eigenvalues with positive real part there is a neighborhood in which all solutions are attracted to the associated center manifold W_c with some exponential rate $\mathcal{O}(\mathrm{e}^{-\beta t})$ for a $\beta > 0$. More precisely, in [**Van89**, Theorem 5.17] it is shown that in this case there are strictly positive constants C and β, such that for all x_0 in a neighborhood of the center manifold there is a $t_0 \in \mathbb{R}$ and a $x_c \in W_c$ such that

$$\|x(t, x_0) - x(t - t_0, x_c)\| \leq C\mathrm{e}^{-\beta t}.$$

As a consequence the stability of bifurcating solutions is solely determined by the reduced ODE on the center manifold. ⌋

Similarly, center manifolds can be defined for discrete dynamical systems.

Example 3.2.7. We consider the discrete dynamical system

$$x_{n+1} = x_n + x_n y_n \qquad y_{n+1} = \lambda y_n - x_n^2$$

with $0 < \lambda < 1$. We find $E_c = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. In order to compute the center manifold we make the ansatz

$$y = h(x) = ax^2 + bx^3 + \mathcal{O}(x^4)$$

and find

$$a(x + x(ax^2 + \ldots))^2 + b(x + x(ax^2 + \ldots))^3 + \ldots = \lambda(ax^2 + bx^3 + \ldots) - x^2$$

which yields $a = -\frac{1}{1-\lambda}$ and $b = 0$. Hence, we have

$$y = h(x) = -\frac{x^2}{1 - \lambda} + \mathcal{O}(x^4)$$

and find for the reduced equation

$$x_{n+1} = x_n - \frac{x_n^3}{1-\lambda} + \mathcal{O}(x_n^5) = x_n \left(1 - \frac{x_n^2}{1-\lambda} + \mathcal{O}(x_n^4)\right).$$

Therefore, $x = 0$ is asymptotically stable in the reduced equation which implies the asymptotic stability of the origin $(x, y) = (0, 0)$ in the full system, similarly to the previous Remark 3.2.6. ⌋

Example 3.2.8. (Saddle-Node bifurcation on center manifold) We consider

$$\dot{x} = \varepsilon + x^2 + y^2 \quad \text{and} \quad \dot{y} = -y + x^2 \tag{3.8}$$

with small ε. For $\varepsilon = 0$ we have the fixed point $(x, y) = (0, 0)$ with eigenvalues $0, -1$ with neutral direction $(1, 0)$. Thus we expand the center manifold as $y = h(x, \varepsilon) = ax^2 + bx\varepsilon + c\varepsilon^2 + \ldots$, which yields $a = 1$, $b = -2$, $c = 2$ and hence $\dot{x} = \varepsilon + x^2 + \mathcal{O}(x^4)$ as reduced equation. We have a saddle-node bifurcation with two fixed points $-\sqrt{-\varepsilon}$ (stable) and $\sqrt{-\varepsilon}$ (saddle) for $\varepsilon < 0$ and no fixed point for $\varepsilon > 0$. Thus we do not actually need h. See Figure 3.12 for the phase portrait. ⌋

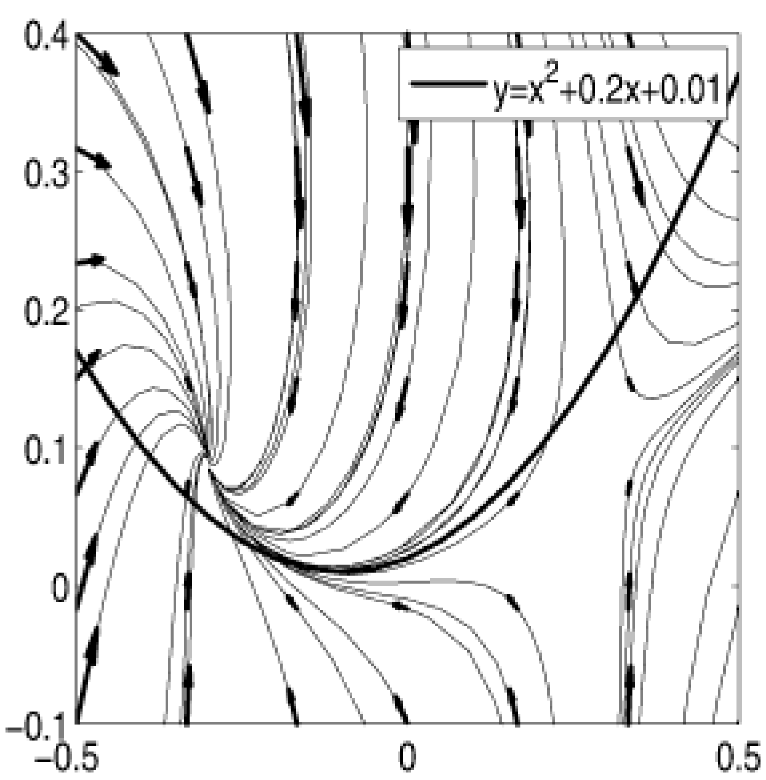


Figure 3.12. Saddle-node bifurcation on center manifold; $\varepsilon = -0.1$. .

As already said, some parts of the center manifold theorem, cf. Theorem 3.2.1, namely the invariance, the existence, and the Lipschitz-continuity of the center manifold will be proven in §13.1. There we will explain that center manifold theory is not restricted at all to the finite-dimensional situation. In Part IV it is used for the construction of bifurcating spatially periodic solutions of pattern forming systems, but also in the construction of traveling wave solutions in unbounded cylindrical domains.

3.3. The Hopf bifurcation

In case that two complex conjugate eigenvalues cross the imaginary axis, the analysis of the system can be reduced to the analysis of the associated two-dimensional center manifold. For the ODE on the center manifold, however, a large number of coefficients have to be computed, namely six coefficients for the quadratic terms and eight coefficients for the cubic terms. Hence, at a first glance a big zoo of possible dynamics can be expected. However, this is not true. By normal form transformations the problem can be reduced in polar coordinates to

$$\dot{r} = \nu_1 r + \nu_2 r^3 + \dots \quad \text{and} \quad \dot{\phi} = 1 + \dots, \tag{3.9}$$

with $\nu_1, \nu_2 \in \mathbb{R}$, i.e., to the computation of two efficient coefficients. Ignoring the higher order terms this system has already been discussed in Example 3.1.4. Therefore, in case that two complex conjugate eigenvalues cross the imaginary axis, under some non-degeneracy condition, always time-periodic solutions occur, either as sub- or supercritical bifurcation.

Theorem 3.3.1. *Consider $\dot{x} = A_\mu x + g(x)$ with $x(t) \in \mathbb{R}^d$ and $\|g(x)\| = \mathcal{O}(\|x\|^2)$ for $x \to 0$. Assume that for $\mu = \mu_0$ the matrix A_μ possesses two eigenvalues $\lambda_\pm = \pm \mathrm{i}\omega$ with $\omega \neq 0$ and that all other eigenvalues possess strictly negative real part. Furthermore assume that $\frac{\mathrm{d}\,\mathrm{Re}\,\lambda_\pm}{d\mu}|_{\mu=\mu_0} \neq 0$. If $\nu_2 \neq 0$ in (3.9), or more precisely $\gamma_r \neq 0$ in (3.13) below, then a one parametric family of periodic solutions bifurcates from $x = 0$ at $\mu = \mu_0$. The period of the bifurcating solutions is $2\pi/\omega + \mathcal{O}(|\mu - \mu_0|)$ and their amplitude is of order $\mathcal{O}(|\mu - \mu_0|^{1/2})$.*

Proof. For the somewhat lengthy proof we introduce the new bifurcation parameter $\varepsilon = \mu - \mu_0$ and extend the ODE system with $\dot{\varepsilon} = 0$. Then we apply the center manifold theorem and reduce the full system to a system on the three-dimensional center manifold associated to the eigenvalues $\lambda_\pm$ and the variable ε.

On the center manifold M_c for arbitrary coordinates $(y, z) \in \mathbb{R}^2$ the reduced system can be written as

$$\begin{aligned} \dot{y} &= a_{11}y + a_{12}z + a_{120}y^2 + a_{111}yz + a_{102}z^2 + a_{130}y^3 + a_{121}y^2 z \\ &\qquad + a_{112}yz^2 + a_{003}z^3 + \mathcal{O}(\varepsilon^2(|x|+|y|) + |y|^4 + |z|^4), \\ \dot{z} &= a_{21}y + a_{22}z + a_{220}y^2 + a_{211}yz + a_{202}z^2 + a_{230}y^3 + a_{221}y^2 z \\ &\qquad + a_{212}yz^2 + a_{203}z^3 + \mathcal{O}(\varepsilon^2(|x|+|y|) + |y|^4 + |z|^4), \\ \dot{\varepsilon} &= 0, \end{aligned} \tag{3.10}$$

where the values of the real-valued coefficients $a_\cdot = a_\cdot(\varepsilon)$ depends on our choice of basis. The only restriction so far is that the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

possesses the eigenvalues

$$\lambda_{\pm}(\mu) = \pm \mathrm{i}\omega + \mathcal{O}(\varepsilon).$$

At a first view, all kinds of dynamics seem to be possible. However, the system can be simplified heavily with a so called normal form transform. In order to do so we diagonalize (3.10) and obtain

$$\begin{aligned} \dot{c}_1 &= \mathrm{i}\omega c_1 + \alpha^1_{110}\varepsilon c_1 + \alpha^1_{101}\varepsilon c_{-1} \\ &\qquad + \alpha^1_{020}c_1^2 + \alpha^1_{011}c_1c_{-1} + \alpha^1_{002}c_{-1}^2 + \dots, \\ \dot{c}_{-1} &= -\mathrm{i}\omega c_{-1} + \alpha^2_{110}\varepsilon c_1 + \alpha^2_{101}\varepsilon c_{-1} \\ &\qquad + \alpha^2_{020}c_1^2 + \alpha^2_{011}c_1c_{-1} + \alpha^2_{002}c_{-1}^2 + \dots, \\ \dot{\varepsilon} &= 0, \end{aligned} \tag{3.11}$$

with $c_1 = \overline{c_{-1}}$ and coefficients $\alpha_{\cdot} \in \mathbb{C}$. The idea of the normal transform is to eliminate all terms which are not in resonance with the linear ones. As an example consider c_{-1}^2 in the equation for c_1. It oscillates as $\mathrm{e}^{-2i\omega t}$ and is therefore not in resonance with c_1, which oscillates as $\mathrm{e}^{i\omega t}$. Therefore, this c_{-1}^2 term be eliminated. With this heuristic argument the only terms which remain in the equation for c_1 are those of the form $\varepsilon^m c_1^n c_{-1}^{n-1}$, and in the equation for c_{-1} those of the form $\varepsilon^m c_1^{n-1} c_{-1}^n$. This heuristic argument can be made rigorous by making a number of near identity changes of variables.

3.3.1. Normal form transforms. We consider the autonomous system

$$\dot{x} = Ax + f(x)$$

for $x(t) \in \mathbb{R}^d$, with $A \in \mathbb{R}^{d \times d}$, and

$$f(x) = f_2(x) + f_3(x) + f_4(x) + \dots,$$

with $f_m(kx) = k^m f_m(x)$ for all $k \geq 0$, i.e., f_m is a vector in $\mathbb{R}^d$ whose entries are homogeneous polynomials of degree m in the variables $x_1, \dots, x_d$. Hence,

$$f_m = \begin{pmatrix} f_{m1} \\ \vdots \\ f_{md} \end{pmatrix}$$

is an element of the vector space

$$V_m = \left\{ u = \begin{pmatrix} \sum_{m_1+\dots+m_d=m} \alpha^1_{m_1\dots m_d} x_1^{m_1} \cdot \ldots \cdot x_d^{m_d} \\ \vdots \\ \sum_{m_1+\dots+m_d=m} \alpha^d_{m_1\dots m_d} x_1^{m_1} \cdot \ldots \cdot x_d^{m_d} \end{pmatrix} : \alpha^j_{m_1\dots m_d} \in \mathbb{R} \right\},$$

the space of vector valued homogeneous polynomials of degree m in the variables $x_1, \dots, x_d$.

We look for a near identity change of variables which allows us to eliminate as many terms as possible in order to make the system as simple as possible. Therefore, we make the ansatz

$$x = y + h(y),$$

where

$$h(y) = h_2(y) + h_3(y) + h_4(y) + \dots$$

with $h_m \in V_m$. We obtain

$$\dot{x} = \dot{y} + \frac{\partial h}{\partial y}\dot{y} = A(y + h(y)) + f(y + h(y)),$$

and therefore

$$\begin{aligned}\dot{y} &= \left(1 + \frac{\partial h}{\partial y}\right)^{-1} [A(y + h(y)) + f(y + h(y))] \\ &= Ay - \frac{\partial h_2}{\partial y}Ay + Ah_2(y) + f_2(y) + \mathcal{O}(\|y\|^3).\end{aligned}$$

In order to eliminate all quadratic terms in f_2 we have to find an h_2 such that

$$-\frac{\partial h_2}{\partial y}Ay + Ah_2(y) + f_2(y) = 0.$$

With the above interpretation of h_2 as an element of the vector space V_2,

$$(L_A h_2)(y) = -\frac{\partial h_2}{\partial y}Ay + Ah_2(y)$$

is a linear map of V_2 into itself which acts linearly on the coefficients $\alpha^m_{m_1\dots m_d}$.

Similarly, in order to eliminate terms of order m we have to solve the linear system

$$-\frac{\partial h_m}{\partial y}Ay + Ah_m(y) + \widetilde{f}_m(y) = 0,$$

where $\widetilde{f}_m$ represents the nonlinear terms of degree m after the application of the transformations h_2 to h_{m-1}.

For our purposes it is sufficient to restrict ourselves to the case of a diagonal A, i.e., $A = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$. Then in the space V_m the linear map L_A possesses the eigenvectors $y_1^{m_1} \cdot \ldots \cdot y_d^{m_d} \mathrm{e}_j$, where e_j is the j^{th} unit vector of $\mathbb{R}^d$. The associated eigenvalues are given by $\mu = \sum_{k=1}^d m_k\lambda_k - \lambda_j$. In order to see this, we consider the j^{th} component

$$\sum_{k=1}^d \frac{\partial h_{mj}}{\partial y_k}\lambda_k y_k - \lambda_j h_{mj} = \mu h_{mj}$$

of the eigenvalue equation $L_A h_m = \mu h_m$. Inserting the above eigenvectors shows the statement. Therefore, $L_A h_m = g_m$ can be solved w.r.t. h_m in all eigenspaces with eigenvalue $\mu \neq 0$. We found

term	non-resonance condition	can be eliminated
$\alpha_{20}c_1^2$	$-2\lambda_1 - 0\lambda_2 + \lambda_1 = -2\mathrm{i} + \mathrm{i} \neq 0$	yes
$\alpha_{11}c_1c_{-1}$	$-0\lambda_1 - 2\lambda_2 + \lambda_1 = -2(-\mathrm{i}) + \mathrm{i} \neq 0$	yes
$\alpha_{02}c_{-1}^2$	$-0\lambda_1 - 2\lambda_2 + \lambda_1 = -2(-\mathrm{i}) + \mathrm{i} \neq 0$	yes
$\alpha_{30}c_1^3$	$-3\lambda_1 - 0\lambda_2 + \lambda_1 = -3\mathrm{i} + \mathrm{i} \neq 0$	yes
$\alpha_{21}c_1^2c_{-1}$	$-2\lambda_1 - 1\lambda_2 + \lambda_1 = -2\mathrm{i} - (-\mathrm{i}) + \mathrm{i} = 0$	no
$\alpha_{12}c_1c_{-1}^2$	$-1\lambda_1 - 2\lambda_2 + \lambda_1 = -\mathrm{i} - 2(-\mathrm{i}) + \mathrm{i} \neq 0$	yes
$\alpha_{03}c_{-1}^3$	$-0\lambda_1 - 3\lambda_2 + \lambda_1 = -3(-\mathrm{i}) + \mathrm{i} \neq 0$	yes

Table 1. Non-resonance conditions for the terms in the first equation of (3.11).

Lemma 3.3.2. *Assume* $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. *Then in order to eliminate the term* $y_1^{m_1} \cdot \ldots \cdot y_d^{m_d}$ *in the* j^{th} *equation we need the non-resonance condition*

$$\sum_{k=1}^{d} m_k\lambda_k - \lambda_j \neq 0. \tag{3.12}$$

3.3.2. Continuation of the proof of Theorem 3.3.1. We now make a normal form transform for (3.11). In Table 1 we list the various terms in the first equation of (3.11) and their non-resonance conditions. Hence, after the transformation we obtain for the new variables $b_j = c_j + \mathcal{O}(|\varepsilon|(|c_1| + |c_{-1}|) + |c_1|^2 + |c_{-1}|^2)$ that

$$\begin{aligned}
\dot{b}_1 =& \mathrm{i}\omega b_1 + \beta_1\varepsilon b_1 + \gamma_1 b_1^2 b_{-1} + \mathcal{O}(|\varepsilon^2|(|b_1| + |b_{-1}|) + |b_1|^4 + |b_{-1}|^4), \\
\dot{b}_{-1} =& -\mathrm{i}\omega b_{-1} + \beta_{-1}\varepsilon b_{-1} + \gamma_{-1} b_1 b_{-1}^2 + \mathcal{O}(|\varepsilon^2|(|b_1| + |b_{-1}|) + |b_1|^4 + |b_{-1}|^4),
\end{aligned}$$

with $b_1 = \overline{b_{-1}}$, $\beta_1 = \overline{\beta_{-1}} = \beta_r + \mathrm{i}\beta_i$, and $\gamma_1 = \overline{\gamma_{-1}} = \gamma_r + i\gamma_i$, where β_r, β_i, γ_r, $\gamma_i \in \mathbb{R}$. Introducing polar coordinates $b_1 = re^{\mathrm{i}\phi}$ gives the system

$$\begin{array}{rcl}
\dot{r} &=& \beta_r\varepsilon r + \gamma_r r^3 + \mathcal{O}(\varepsilon^2 r + r^4), \\
\dot{\phi} &=& \omega + \beta_i\varepsilon + \gamma_i r^2 + \mathcal{O}(\varepsilon^2 + r^3).
\end{array} \tag{3.13}$$

Hence, we have a system which can be analyzed for small ε. Ignoring the higher order terms we have $r = \mathcal{O}(\sqrt{\varepsilon})$ for the bifurcating time-periodic solutions. For the scaled variable $\widetilde{r}$ defined by $r = \sqrt{\varepsilon}\widetilde{r}$ we find the approximate time-periodic solution $\widetilde{r}_0^2 = -\beta_r/\gamma_r$. Depending on the sign of γ_r we have a sub- or supercritical bifurcation of time-periodic solutions. In order to prove the persistence of these solutions under the neglected $\mathcal{O}(\varepsilon^2)$-terms we construct the associated Poincaré map Π_ε for which the periodic solution is a fixed point. The fixed point is therefore a zero of

$$F(\widetilde{r}(0), \varepsilon) = \varepsilon^{-1/2}(\Pi_\varepsilon(\widetilde{r}(0)) - \widetilde{r}(0)).$$

We have $F(\widetilde{r}_0(0), 0) = 0$ and $\partial_1 F(\widetilde{r}_0(0), 0) = -4\pi\beta_r\omega^{-1} \neq 0$ such that the implicit function theorem can be applied and a fixed point $\widetilde{r}(0) = \widetilde{r}_\varepsilon(0)$ of

Π_ε exists for $\varepsilon > 0$, too. Associated to this fixed point are periodic solutions of (3.13), (3.11), and finally of (3.10). Therefore, we are done. □

3.3.3. An example and further remarks. In applications very often the parameter space has more than one dimension, i.e., the parameters are given by $p \in \mathbb{R}^m$. In principle, this situation can be handled in the same way as above by varying the parameters individually. Generically, the set $\{p \in \mathbb{R}^m : F(x(p), p) = 0, \partial_x F(x(p), p) = 0\}$ of possible bifurcation values becomes a $(m-1)$-dimensional manifold. Here we consider the following example of a 2-parameter bifurcation/stability diagram.

Example 3.3.3. Bifurcation diagram for a (toy) problem from chemistry. The system

$$\dot\alpha = \mu - \alpha(\kappa + \beta^2), \qquad \dot\beta = -\beta + \alpha(\kappa + \beta^2) \tag{3.14}$$

serves as a (drastically reduced) model for so called cubic autocatalysis in chemistry. Here, α, β are concentrations (hence $\alpha, \beta \geq 0$) of some substances and the parameters $\mu, \kappa > 0$ are some reaction rates. Unlike other reagents, here represented by α, that participate in the chemical reaction, a catalyst, here represented by β, is not consumed by the reaction itself. The above model describes a situation where the so called educt α is supplied into the system at constant rate μ and converted into β with rate $\kappa + \beta^2$. Thus, the catalyst β catalyses its own production, hence the name autocatalysis.

The unique fixed point of (3.14) is given by

$$(\alpha^*, \beta^*) = (\mu/(\kappa + \mu^2), \mu),$$

with the associated linearization

$$A = \begin{pmatrix} -(\mu^2 + \kappa) & -2\mu^2/(\kappa + \mu^2) \\ \mu^2 + \kappa & (\mu^2 - \kappa)/(\mu^2 + \kappa) \end{pmatrix}. \tag{3.15}$$

First, we discuss the eigenvalues of A which are given in terms of $p = \operatorname{trace} A$ and $q = \det A$ by

$$\lambda_{1,2} = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \tag{3.16}$$

This associated bifurcation/stability diagram is plotted in Figure 3.13. The meaning of this diagram is as follows. Starting, e.g., with p, q in the **sn** regime ($p < 0$ and $0 < q < p^2/4$) and crossing, e.g., the line $q = p^2/4$, the fixed point changes type from **sn** to **sf**, cf. (3.16). Next, depending on the nonlinearity, we may expect a Hopf bifurcation when crossing the line $p = 0, q > 0$, which is therefore called Hopf line. The point $(p, q) = (0, 0)$ is called Bogdanov-Takens or co-dimension-2 point since two parameters are needed to describe the possible bifurcations in its neighborhood.

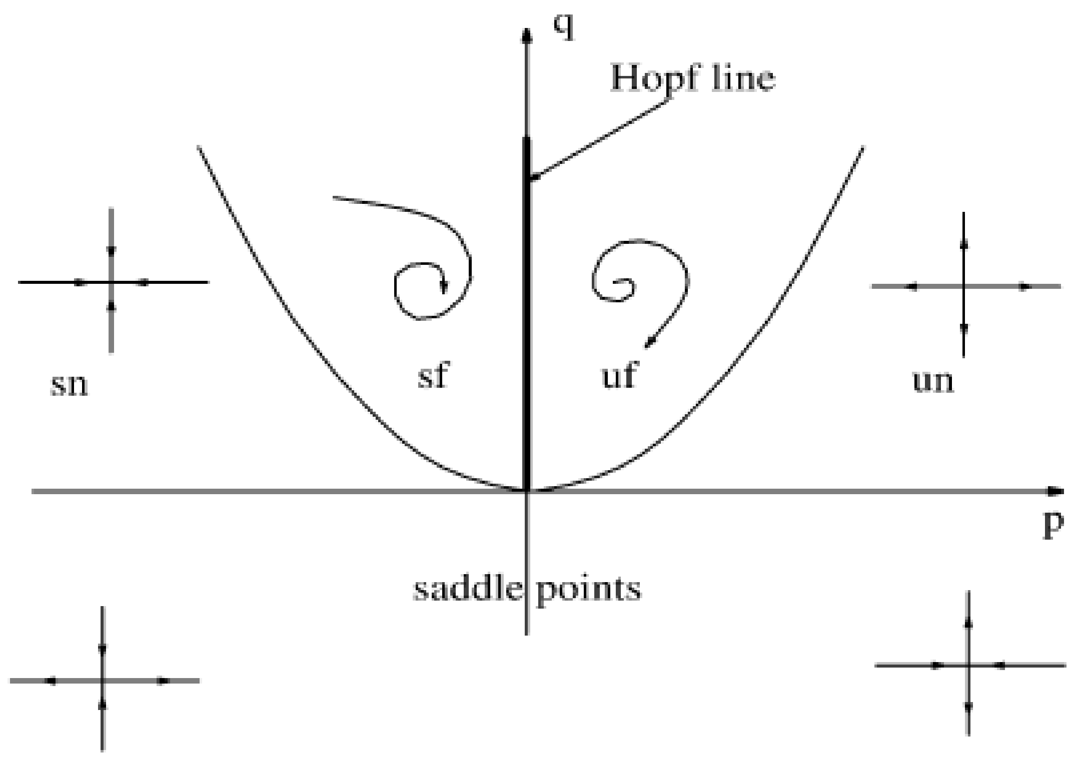


Figure 3.13. Bifurcation diagram for $\dot{x} = Ax$ in terms of the trace p and the determinant q of A. The abbreviations stand for: **sn**: stable node, two stable real eigenvalues; **sf**: stable focus, two stable conjugate complex eigenvalues; **uf**: unstable focus, two unstable conjugate complex eigenvalues; **un**: unstable node, two unstable real eigenvalues; **saddle points:** two real eigenvalues, one stable, the other unstable.

The transfer of Figure 3.13 to the fixed point (α_0, β_0) and to the parameters μ, κ yields to the solution of a number of algebraic equations. For instance, the Hopf line is given by solving the 4th order equation

$$\operatorname{trace} A = -(\mu^4 - (1-2\kappa)\mu^2 + \kappa/(1+\kappa))/(\mu^2 + \kappa) = 0,$$

hence

$$\mu_{1,2}(\kappa) = \frac{1}{\sqrt{2}} \left((1-2\kappa) \pm (1-8\kappa)^{1/2} \right)^{1/2}.$$

In summary, we obtain the bifurcation diagram plotted in Figure 3.14, while Figure 3.15 shows two selected phase portraits. We will come back to such systems in Chapter 9. ⌋

Remark 3.3.4. Besides the analytical study of bifurcating branches close to bifurcation, i.e., the analysis of the reduced equation, there is the big field of numerical path following (or continuation) and bifurcation analysis. The basic idea of continuation is as follows. Given a solution $(x_0, \mu_0) \in \mathbb{R}^2$ of $f(x, \mu) = 0$ with $\partial_x f(x_0, \mu_0) \neq 0$ we choose a small $\delta > 0$, let $\mu = \mu_0 + \delta$, and use the Newton scheme to compute $x(\mu)$. In detail, we use the iteration

$$x_{n+1} = x_n - (\partial_x f(x_n, \mu))^{-1} f(x_n, \mu)$$

with starting point x_0. The scheme converges for $\delta > 0$ sufficiently small and we set $x(\mu) = \lim_{n\to\infty} x_n$. Replacing (x_0, μ_0) by $(x(\mu_0 + \delta), \mu_0 + \delta)$ we can start again and compute solutions $x = x(\mu)$ until $\partial_x f(x(\mu), \mu) = 0$. In case of $\partial_\mu f(x_0, \mu_0) \neq 0$ we can interchange the role of x and μ and obtain a solution

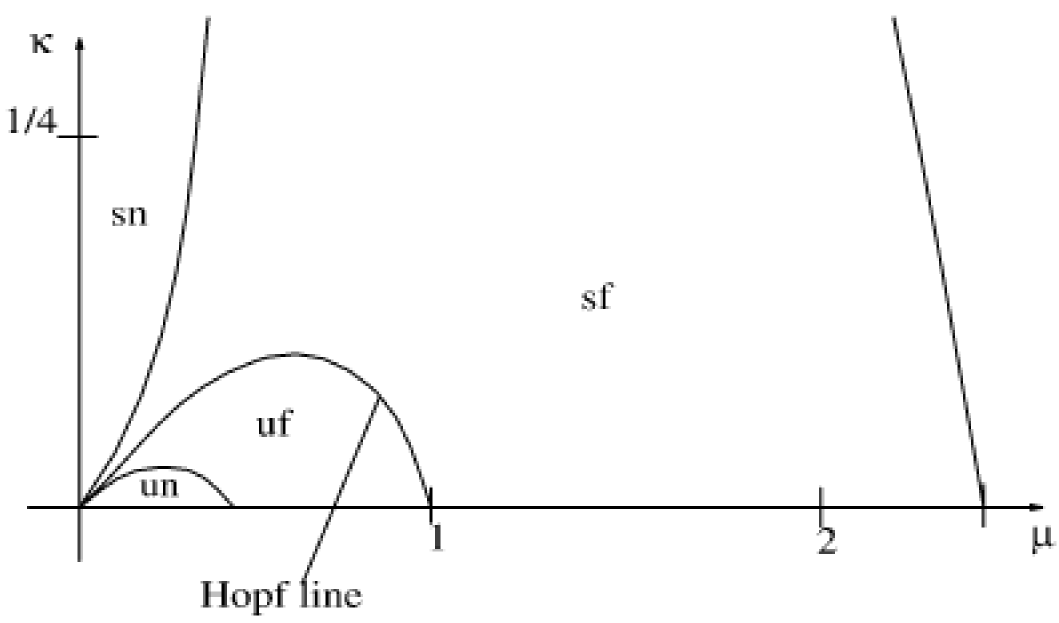


Figure 3.14. The bifurcation diagram for (3.14) is to be read as follows. If, e.g., we fix $\kappa = 0.1$ then we cross the Hopf line at approximately $\mu_1 \approx 0.41$ and $\mu_2 \approx 0.77$. At these lines Hopf bifurcations can be expected. Path following methods may allow us to follow the family of periodic solutions in the parameter plane.

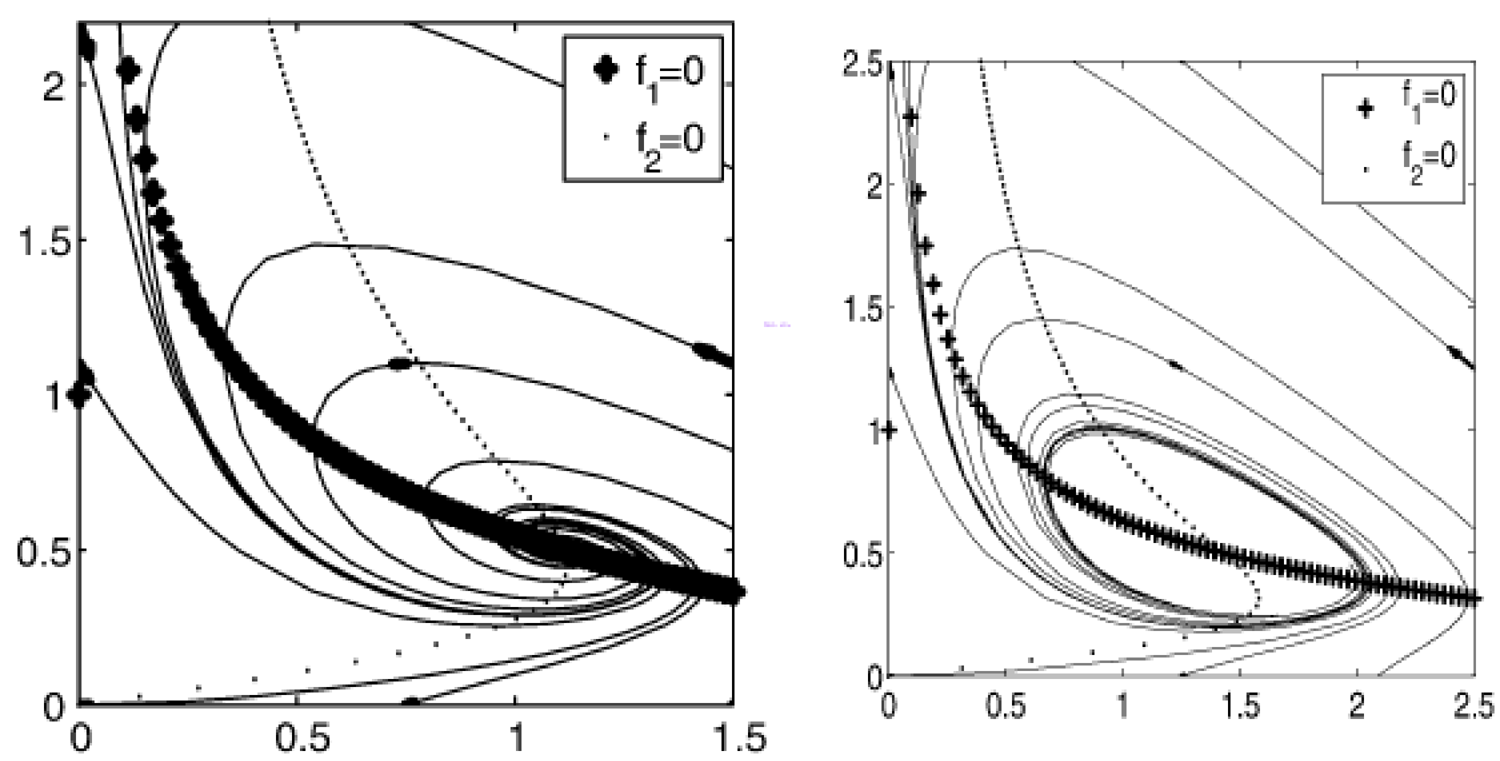


Figure 3.15. Phase portraits for $(\mu, \kappa) = (0.5, 0.2)$ (left) and $(\mu, \kappa) = (0.5, 0.1)$ (right)

$\mu = \mu(x)$, i.e., we have a smooth curve $(x, \mu) = (x, \mu)(s)$ parameterized with s until simultaneously $\partial_x f(x(s), \mu(s)) = \partial_\mu f(x(s), \mu(s)) = 0$.

There are variants of this idea [**Kel77, Kuz04, Doe07, Sey10**] which automatically allow the continuation of branches around folds and beyond bifurcation points, the detection and localization of bifurcation points, and branch switching at bifurcation points. One standard method is so called (pseudo-)arclength continuation, which is implemented in the package AUTO, [**Doe07, Dea16**], see also XPPAUT, [**Erm02**]. Many of these methods can also be applied to bifurcation problems in PDEs and are important tools there. A recent package specifically designed for elliptic systems in two space dimensions is `pde2path`, [**UWR14, DRUW14**]. ⌋

3.4. Routes to chaos

The chapter is closed by sketching two routes of bifurcations to chaotic behavior in dissipative systems, namely period-doubling, which is based on an infinite series of local bifurcations, and homoclinic explosion, which is a so called global bifurcation. There are many other routes to chaotic behavior, but we will only comment on one of them, namely the Ruelle-Takens scenario.

The theory of turbulence developed by Landau [**LL91**] in 1944 is based on the assumption that more and more pairs of complex conjugate eigenvalues cross the imaginary axis. This route to chaos is called the Landau-Hopf scenario. In 1971 Ruelle and Takens [**RT71**] showed that these infinitely many unstable eigenvalues are not necessary for the occurrence of chaos. The scenario starts with a stable fixed point and provides a very short route to chaotic behavior only using local bifurcations. The first bifurcation is a Hopf bifurcation leading to time-periodic solutions. Then the time-periodic solution becomes unstable via a pair of complex-conjugate Floquet multipliers crossing the unit circle leading to quasi-periodic solutions. The next bifurcation leads to a three-dimensional invariant torus where nearby chaotic behavior can be found. This route to chaos plays a certain role in hydrodynamical applications, but now we focus on the route with infinitely many period-doublings.

3.4.1. Period-doubling. Period-doubling is realized in nature in a number of systems, as cardiac diseases, leaking water-taps, laser dynamics, and various chemical reactions. It occurs if a periodic orbit becomes unstable and a stable periodic solution with roughly the double period occurs, and if this repeats under further increase (or decrease) of some parameter, see Figure 3.16.

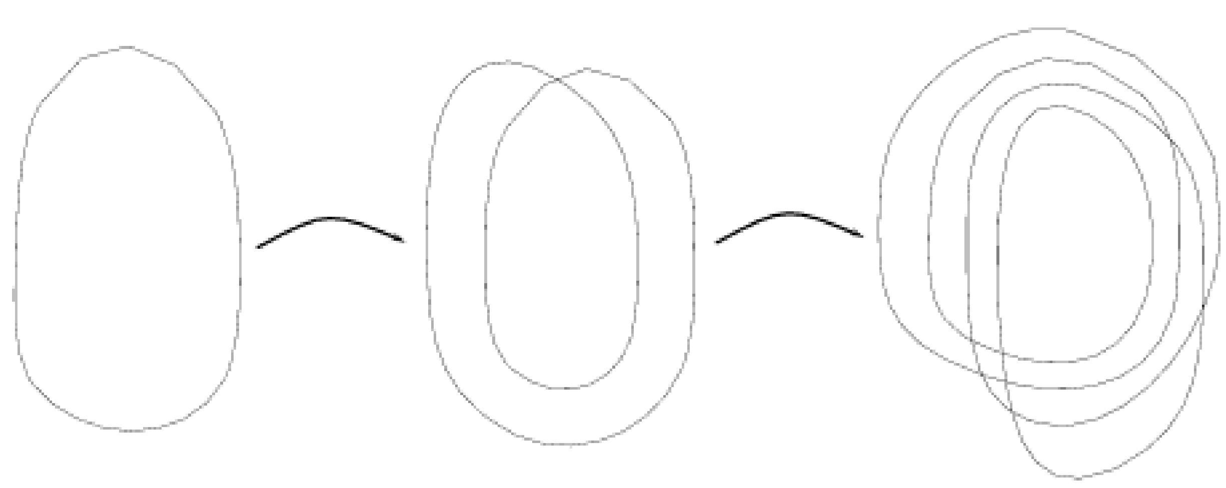


Figure 3.16. Sequence of period-doublings for a periodic solution

For the analysis of this phenomenon we consider the associated Poincaré map Π. The fixed point of Π, which is associated to the periodic orbit, becomes unstable via a real Floquet multiplier crossing the unit circle at

-1. The fixed point becomes also unstable for the second iterate of the Poincaré map Π^2, but now via a real Floquet multiplier crossing the unit circle at 1. On the center manifold associated to the Floquet multiplier -1 we have the following situation. If

$$\Pi x = -x + \alpha x^2 + \mathcal{O}(x^3), \quad \text{then} \quad \Pi^2 x = x + \beta x^3 + \mathcal{O}(x^4),$$

such that for Π^2 a pitchfork bifurcation occurs. The two stable bifurcating fixed points for Π^2 corresponds to a 2-periodic solution for Π itself, since for Π no bifurcation of fixed points occurs. Hence, a new periodic orbit with twice the period is bifurcating from the old one. Assuming that this new periodic orbit becomes unstable in the same way and that this procedure goes on and on we finally come to chaotic dynamics. A famous ODE example showing this behavior is by Rössler [**Rös76**].

Example 3.4.1. The Rössler system. Consider the ODE

$$\partial_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -(y+z) \\ x+by \\ b+z(x-a) \end{pmatrix},$$

where typically $b = 0.1$ and $a \in \mathbb{R}$ serves as a bifurcation parameter. Starting from an asymptotically stable periodic orbit for a=4 we find for increasing a a period-doubling sequence, cf. Figure 3.17. See, e.g., [**PJS92**, §3.3] and the references therein for a more detailed introduction to the Rössler system. ⌋

3.4.2. The logistic map. There is a discrete model problem for the period-doubling route to chaos, namely the logistic map

$$x_{n+1} = \mu x_n(1-x_n) = F(x_n)$$

with $\mu \geq 0$ and $x_n \in \mathbb{R}$. We have for the n^{th} iteration $F^n(x) \to -\infty$ for $n \to \infty$ if $x < 0$ or $x > 1$. For $\mu \in [0,4]$ the map F maps the interval $[0,1]$ into itself. In the following we restrict ourselves to values μ and x_0 in these sets. More details can be found in [**Dev89**], including a discussion of chaos in the strict sense of Definition 2.5.2 in the logistic map for parameter values $\mu > 2 + \sqrt{5}$.

The condition $F(x) = \mu x(1-x) = x$ gives the fixed points $x_1^* = 0$ and $x_2^* = 1 - 1/\mu$. At $\mu_0 = 1$ a transcritical bifurcation of fixed points occurs. The linearization around the fixed point x^* is given by $y_{n+1} = F'(x^*)y_n$ where $F'(x^*) = \mu(1-2x^*)$. For $x^* = 1-1/\mu$ we obtain

$$F'(x^*) = \mu(1-2(1-1/\mu)) = 2-\mu.$$

Hence, this fixed point is stable for $\mu \in (1,3)$ and becomes unstable at $\mu_1 = 3$ via some period-doubling. A stable two-periodic solution appears. See Figure 3.18.

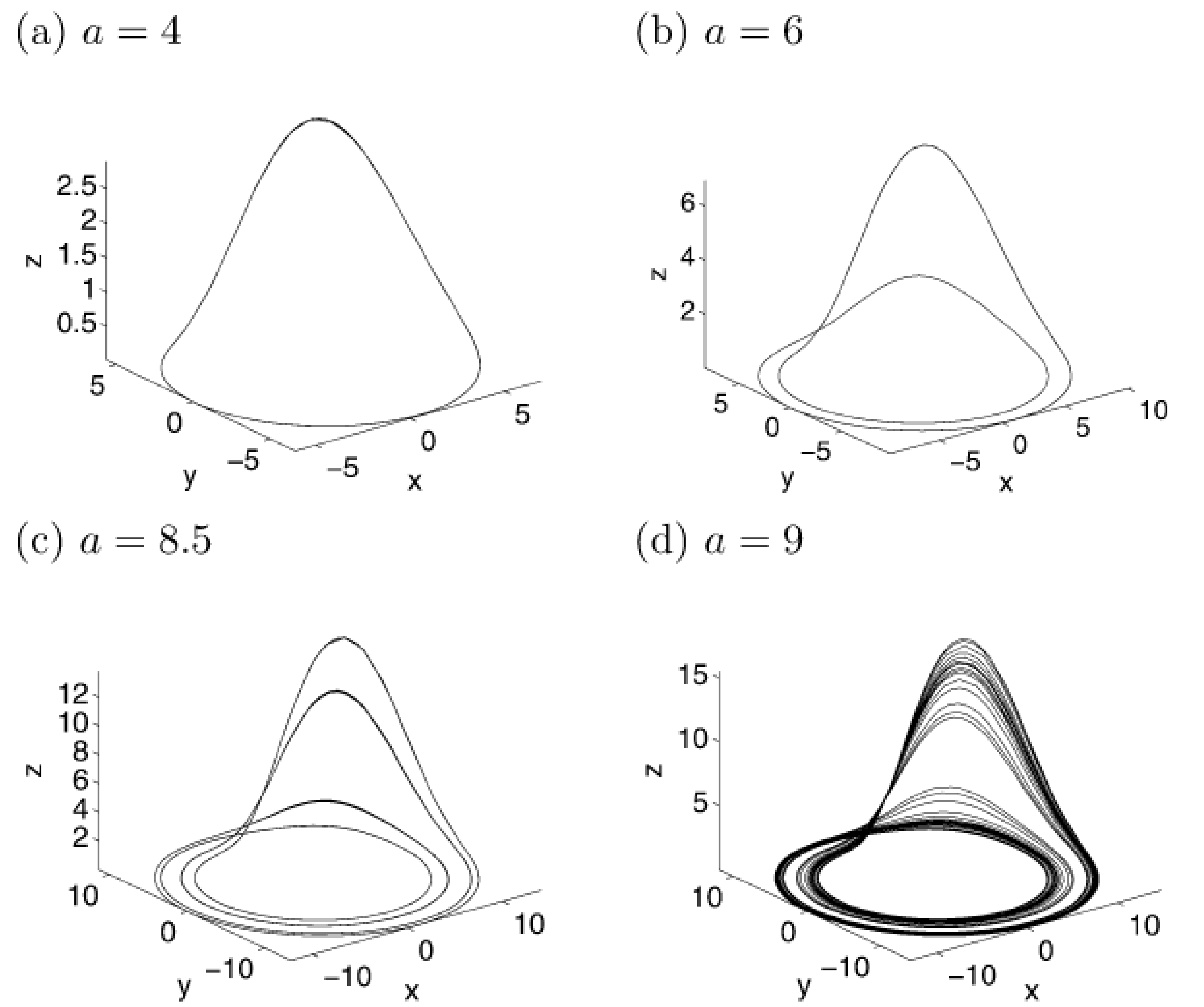


Figure 3.17. Period-doubling in the Rössler system, $b = 0.1$ fixed.

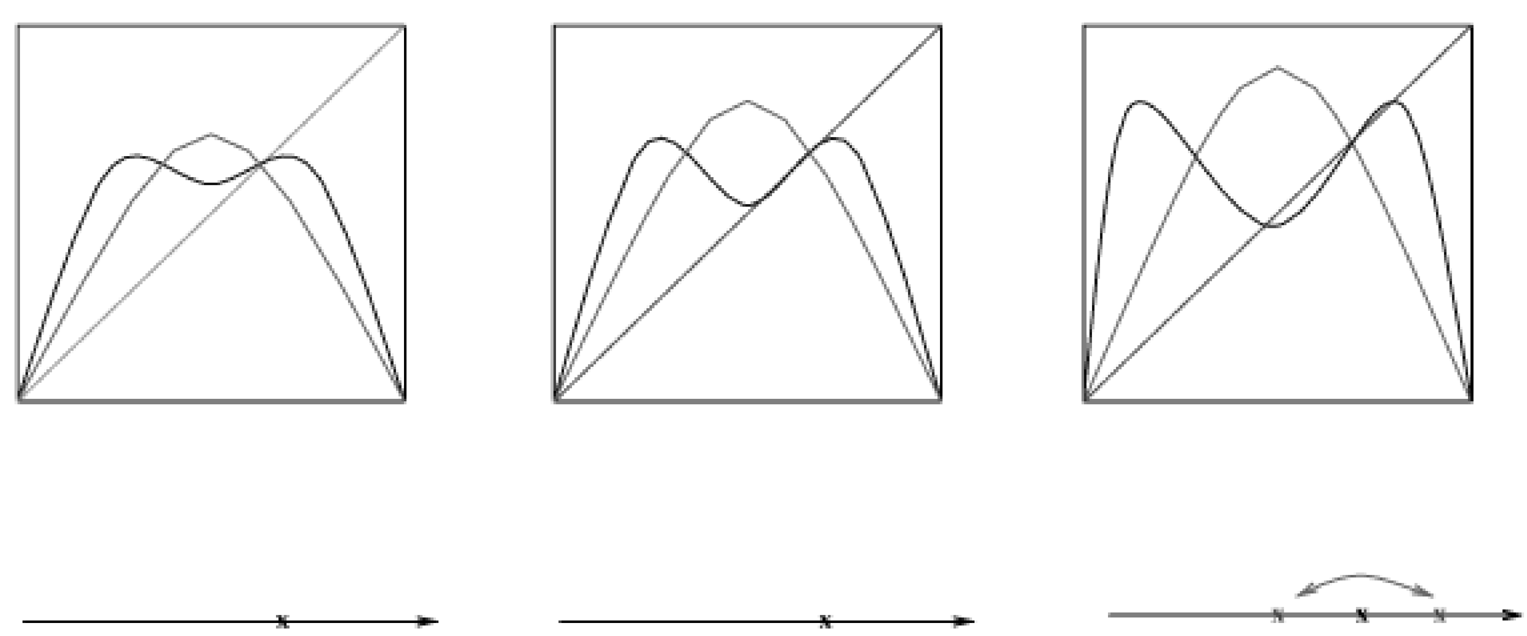


Figure 3.18. The map F^2 for $\mu = 2.9$, $\mu = 3$ and $\mu = 3.1$. A pitchfork bifurcation occurs which corresponds to the occurrence of a two-periodic solution for F.

A further increase of μ yields an instability of the two-periodic solution at a value $\mu = \mu_2$. We find a pitchfork bifurcation for F^4 and so the occurrence of a four-periodic solution for F. A further increase of μ yields an instability of the four-periodic solution at a value $\mu = \mu_3$. We find a pitchfork bifurcation of F^8 and so the occurrence of an eight-periodic solution for F. Interestingly, there is an infinite sequence of such bifurcations and so a further increase of μ yields an instability of the 2^{n-1}-periodic solution at a value $\mu = \mu_n$. We find a pitchfork bifurcation of F^{2^n} and so the occurrence of a 2^n-periodic solution for F.

Even more interestingly, the period-doublings show some asymptotic behavior. It can be proved rigorously by computer-assisted proofs that the limit

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \approx 4.6692,$$

called the Feigenbaum constant, exists, cf. [**CE80**]. As a consequence we have the existence of $\mu_\infty = \lim_{n\to\infty} \mu_n \approx 3.57$, too. A recent overview about the theoretical background of the occurrence of these limits is [**Avi11**]. For most values $\mu > \mu_\infty$ the system exhibits chaotic behavior. In Figure 3.19 the ω-limit set for starting point $x_0 = 1/2$ is plotted as a function over the bifurcation parameter μ. There are isolated regions on the μ-axis where no attractive chaotic behavior occurs, the so called windows of stability. Beginning at $1+\sqrt{8} \approx 3.83$ there is for instance a range of parameters μ with a stable 3-periodic solution. There is a general theory [**Dev89**, §1.10] that for maps from $\mathbb{R}$ to $\mathbb{R}$ solutions of period 3 imply the existence of periodic solutions of every period $m \in \mathbb{N}$, known as the Theorem of Sarkovskii.

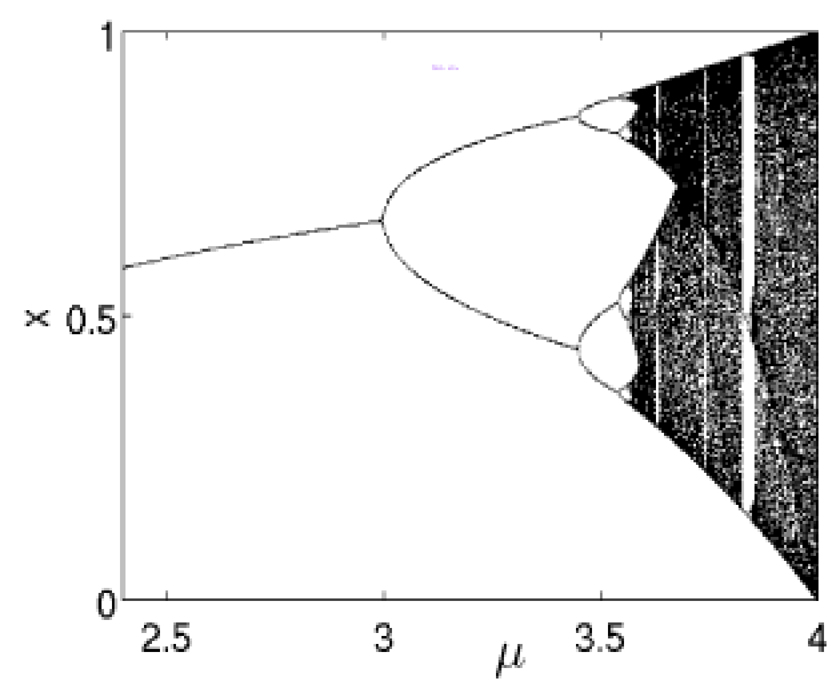


Figure 3.19. The ω-limit set for starting point $x_0 = 1/2$ is plotted as a function over the bifurcation parameter μ. For every fixed μ the iterates $x_N, \ldots x_{N+M}$ with N and M sufficiently large are plotted.

For ODEs the instability occurs in a one-dimensional center manifold where the dynamics can be described via a one-dimensional Poincaré map and hence it can be expected that the route to chaos via period-doublings may occur in ODE systems, too. Another example is the so called chemostat, see Exercise 3.6.

3.4.3. Homoclinic explosion – the Lorenz attractor. The Lorenz attractor [**Lor63, Spa82**] is a famous example of a three-dimensional ODE with chaotic dynamics. It was found by the meteorologist E. Lorenz in 1963 as a lowest order approximation for convection in fluids and is considered as a cartoon weather model. It is given by

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{3.17}$$

with $\sigma = 10$ and $\beta = 8/3$ fixed. Numerical simulations of this simple model for $\rho = 27$ show a complicated irregular dependence of the solutions on the initial conditions and the occurrence of a so called "strange" attractor. A rigorous proof for the occurrence of chaotic dynamics in the Lorenz model (similar to shift dynamics (Σ_2, σ) defined in §2.5.1) has been given [**Tuc02**]. We now explore the route to chaos in a bit more detail.

For any $\sigma, \beta, \rho > 0$ any large enough sphere around $(0, 0, \rho + \sigma)$ is absorbing. This can be shown with the Lyapunov function

$$V(x, y, z) = x^2 + y^2 + (z - \rho - \sigma)^2.$$

With $\alpha = \min\{2\sigma, 2, \beta\}$ we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V &= -2\sigma x^2 - 2y^2 - 2\beta z^2 + 2\beta(\rho + \sigma)z \\ &= -2\sigma x^2 - 2y^2 - \beta(z - \rho - \sigma)^2 - \beta z^2 + \beta(\rho + \sigma)^2 \\ &\leq -\alpha V + \beta(\rho + \sigma)^2. \end{aligned}$$

Hence, for t large enough by Gronwall's inequality we obtain

$$V(t) \leq \frac{2\beta(\rho + \sigma)^2}{\alpha}.$$

By Theorem 2.4.4 there exists the global attractor $\mathcal{A} = \omega(B)$, for which numerical simulations show its geometric complexity. The attractor has a dimension less than three since the divergence of the vector field

$$\partial_{y_1}(\sigma(y_2 - y_1)) + \partial_{y_2}(\rho y_1 - y_2 - y_1 y_3) + \partial_{y_3}(-\beta y_3 + y_1 y_2) = -(\sigma + 1 + \beta)$$

is negative and, therefore, every test volume shrinks to zero for $t \to \infty$. For $\rho = 27$ numerical experiments show a non-integer Hausdorff-dimension of $\mathcal{A}$ of approximately 2.04.

The behavior occurs by a global bifurcation which is called homoclinic explosion. The route to chaos for the Lorenz system is as follows when ρ is increased from 0 to 27, for $\sigma = 10$ and $\beta = 8/3$ fixed. The z-axis is an invariant set, and the origin is a stable fixed point for $\rho < 1$. The linearization of (3.17) around 0 is

$$\nabla f(x, y, z) = \begin{pmatrix} -\sigma & -\sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}.$$

At $(x, y, z) = (0, 0, 0)$ we find the eigenvalues

$$\lambda_{1,2} = -\frac{\sigma+1}{2} \pm \frac{1}{2}\sqrt{(\sigma+1)^2 + 4\sigma(\rho-1)}, \quad \lambda_3 = -\beta,$$

such that at $\rho = 1$ a bifurcation of fixed points occurs which turns out to be a supercritical pitchfork bifurcation. For $\rho > 1$ we have two non-trivial fixed points X_1^*, X_2^* with

$$z = \rho - 1 \quad \text{and} \quad x = y = \pm\sqrt{\beta(\rho-1)}. \tag{3.18}$$

The eigenvalues of $\nabla f(X_{1,2}^*)$ are the roots of

$$p(\lambda) = \lambda^3 + (\sigma + \beta + 1)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\sigma\beta(\rho - 1),$$

From this we find that the fixed points $X_{1,2}^*$ are stable until $\rho = \rho_{\text{Hopf}} \approx 24.74$, where two complex conjugate eigenvalues cross the imaginary axis. It turns out that a subcritical Hopf bifurcation occurs. This means that for $\rho \overset{<}{\approx} \rho_{\text{Hopf}}$ unstable periodic solutions near $X_{1,2}^*$ exist, which shrink to $X_{1,2}^*$ as $\rho \to \rho_{\text{Hopf}}$.

Figure 3.20 gives some numerical illustrations: In general for $\rho \in (1, 24]$ the two parts S_1 and S_2 of the one-dimensional unstable manifold of the origin are connected with the three-dimensional stable manifolds of the fixed points X_1^* and X_2^*. Though it is difficult to see, there is a value $\rho = \rho_{\text{global}} \approx$

(a) $S_{1,2}$ for $\rho = 13$. (b) $S_{1,2}$ for $\rho = 13.91$.

(c) $S_{1,2}$ for $\rho = 24$. (d) A visualization of the attractor.

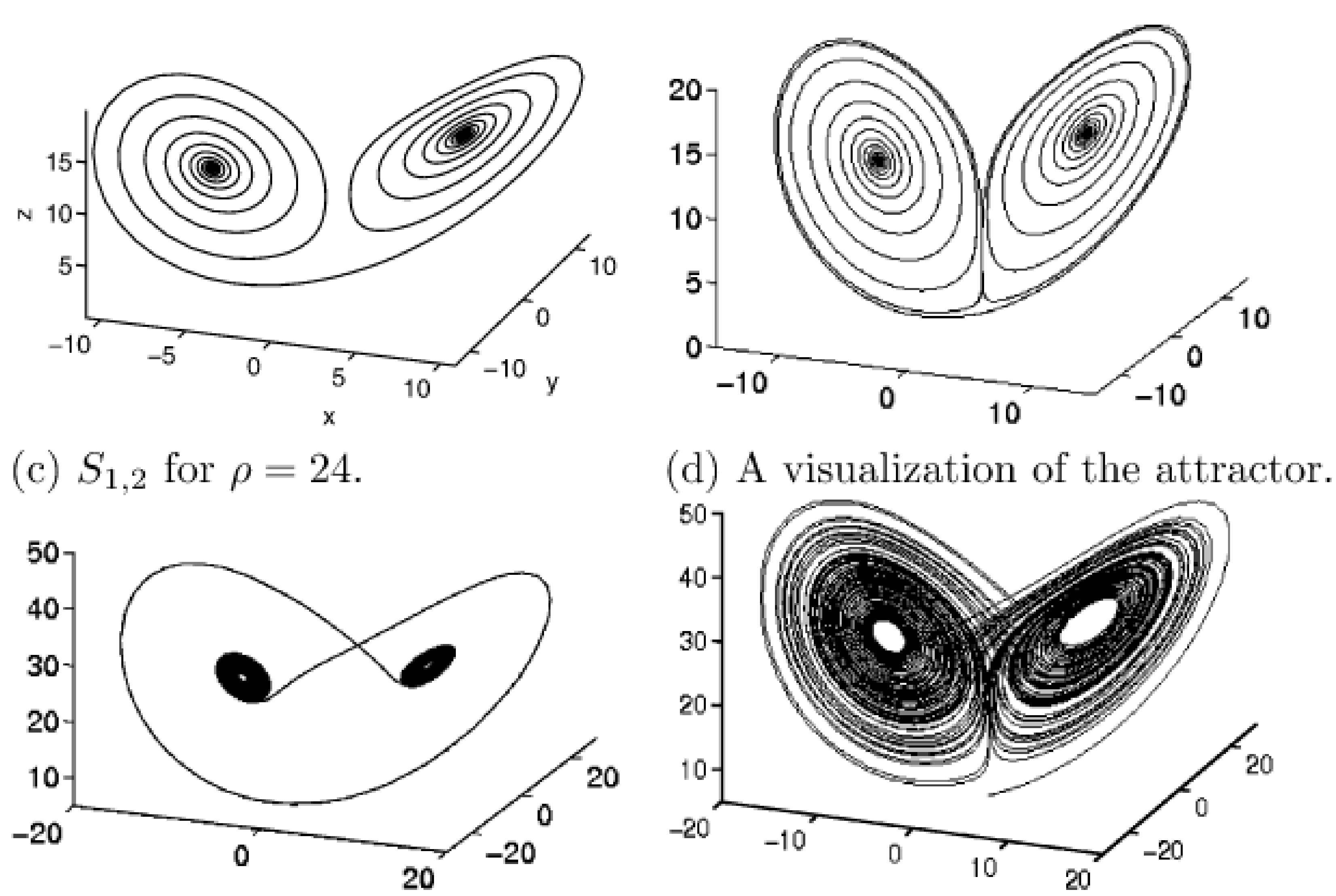


Figure 3.20. The unstable manifolds of the origin in the Lorenz system for different ρ (a)-(c), and the "Lorenz attractor" for $\rho = 27$ (d), visualized by one trajectory.

13.91 where the two parts S_1 and S_2 are connected with the two-dimensional stable manifold of the origin and form two homoclinic connections, see Figure 3.20(b). For $\rho < \rho_{\text{global}}$ the part S_1 connects to X_1^* and S_2 to X_2^*, see Figure 3.20(a), and vice versa for $\rho > \rho_{\text{glob}}$. This behavior is the origin of a so called homoclinic explosion, cf. [**Wig88**], and creates the chaotic behavior in the system. Therefore, chaotic behavior is present in the system already for ρ close to ρ_{global}, but becomes only attractive for larger values of ρ. Figures 3.20(c) and (d)) illustrate the different behavior for $\rho = 24 < \rho_{\text{Hopf}}$ and $\rho = 27$ (Figure 3.20(d)).

An elementary but more detailed introduction to the Lorenz system can be found in [**Str94**, §9], including explanations of simple mechanical and electronic systems able to simulate the Lorenz system, together with applications to send encrypted messages.

Exercises

3.1. For the following ODEs $\dot{x}=f(x)$ determine all fixed points and their stability in dependence of the parameter $\mu\in\mathbb{R}$. What bifurcations occur at what μ?
a) $\dot{x}=\mu+6+4x-x^2$, b) $\dot{x} = 2-\mu+x(\mu-4)+3x^2-x^3$, c) $\dot{x}=\mu+x(\mu-1)-x^2$.

3.2. Compute the non-trivial solutions close to the origin for

$$f(x,y,\varepsilon) = \begin{pmatrix} \varepsilon x - yx^2 - x^4 \\ y + x^2 \end{pmatrix} = 0.$$

3.3. Check the stability of $(x,y) = (0,0)$ for the following systems by calculating the center manifold, cf. [**Wig03**].

(a) $\dot{x} = -xy - x^6$, $\dot{y} = -y + x^2$, (b) $\dot{x} = x^2 y - x^5$, $\dot{y} = -y + x^2$, and

(c) $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ -y_n^3 \end{pmatrix}$.

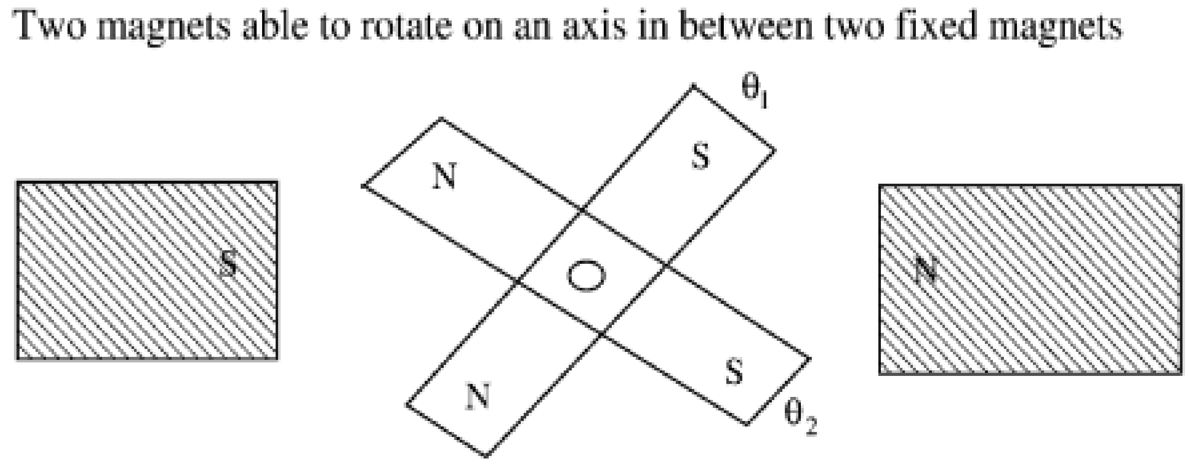
3.4. Which of the following systems has periodic orbits close to $z = (x,y) = (0,0)$? Does Theorem 3.3.1 apply?

$$\begin{aligned}
&(a) \quad \dot{z} = \begin{pmatrix} -\mu^2 & 1 \\ -1 & -\mu^2 \end{pmatrix} z - |z|^2 z, \\
&(b) \quad \dot{z} = \begin{pmatrix} -\mu^2 & 1 \\ -1 & -\mu \end{pmatrix} z - |z|^2 z, \qquad \mu \in \mathbb{R}, \text{ with small } |\mu|. \\
&(c) \quad \dot{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} x^2 - xy \\ xy \end{pmatrix}.
\end{aligned}$$

3.5. This exercise (from [**Str94**]) brings together a number of concepts treated above, namely bifurcation, center manifold calculations, and non-trivial gradient dynamics. The system

$$\begin{aligned}
\dot{\vartheta}_1 &= f_1(\vartheta) := k \sin(\vartheta_1 - \vartheta_2) - \sin\vartheta_1, \\
\dot{\vartheta}_2 &= f_2(\vartheta) := k \sin(\vartheta_2 - \vartheta_1) - \sin\vartheta_2,
\end{aligned}$$

with $\vartheta = (\vartheta_1, \vartheta_2)$ and parameter $k > 0$ describes two rotating magnets between two fixed magnets, in a geometry as follows:



a) The system has exactly nine fixed points in $[-\pi, \pi]^2$ for $k < 1/2$. Determine these and their stability.
b) What bifurcation occurs at $k = 1/2$? What new fixed points emerge?
c) Find a potential V such that $\dot{\vartheta} = -\nabla V(\vartheta)$.
d) Sketch a phase portrait for $0 < k < 1/2$ and a phase portrait for $k > 1/2$.
Remark. Doing c) first and then the rest is a good idea.

3.6. The Chemostat. The chemostat is an industrially used "predator-prey system" to cultivate bacteria. In case of 3 species the system is modeled by

$$\begin{aligned}
\dot{s}(t) &= (1 - s(t)) - \frac{m_1 s(t)}{a_1 + s(t)} x_1(t),\\
\dot{x}_1(t) &= x_1(t) \left(\frac{m_1 s(t)}{a_1 + s(t)} - 1 - \frac{m_2 x_2(t)}{a_2 + x_1(t)} \right),\\
\dot{x}_2(t) &= x_2(t) \left(\frac{m_2 x_1(t)}{a_2 + x_1(t)} - 1 - \frac{m_3 x_3(t)}{a_3 + x_2(t)} \right),\\
\dot{x}_3(t) &= x_3(t) \left(\frac{m_3 x_2(t)}{a_3 + x_2(t)} - 1 \right).
\end{aligned}$$

a) Explain the modeling.
b) Let $\sigma(t) = 1 - s(t) - \sum_{k=1}^{3} x_k(t)$. Show that $\dot{\sigma}(t) = -\sigma(t)$ and use this to prove that the ω-limit set of any solution $(s(t), x_1(t), \dots, x_3(t))$ is contained in

$$\Omega = \{(s, x_1, \dots, x_3) : s + \sum_{k=1}^{3} x_k = 1\}.$$

c) Substitute $s = 1 - \sum_{k=1}^{3} x_k$ into the equations for $\dot{x}_k$ and try to reproduce the period-doubling shown in Figure 3.21.

3.7. Let $Q_c(x) = x^2 + c$. Prove that for all $c < \frac{1}{4}$ there exists a unique $\mu > 1$ such that Q_c is conjugated to $F_\mu(x) = \mu x(1-x)$ through the map $h(x) = \alpha x + \beta$.

3.8. Consider the iteration $x_{n+1} = T_\lambda(x_n)$, where

$$T_\lambda(x) = \lambda \begin{cases} 2x & \text{for } x \in [0, 1/2], \\ 2 - 2x, & \text{for } x \in (1/2, 1]. \end{cases}$$

a) Prove, $x^* = 0$ is an asymptotically stable fixed point for $\lambda \geq 0$ sufficiently small.
b) Compute $T_\lambda^2 = T_\lambda \circ T_\lambda$ and find graphically the 2-periodic solutions, i.e., solve $T_\lambda^2(x) = x$ by finding the intersection points of the functions $x \mapsto T_\lambda^2(x)$ and $x \mapsto x$.

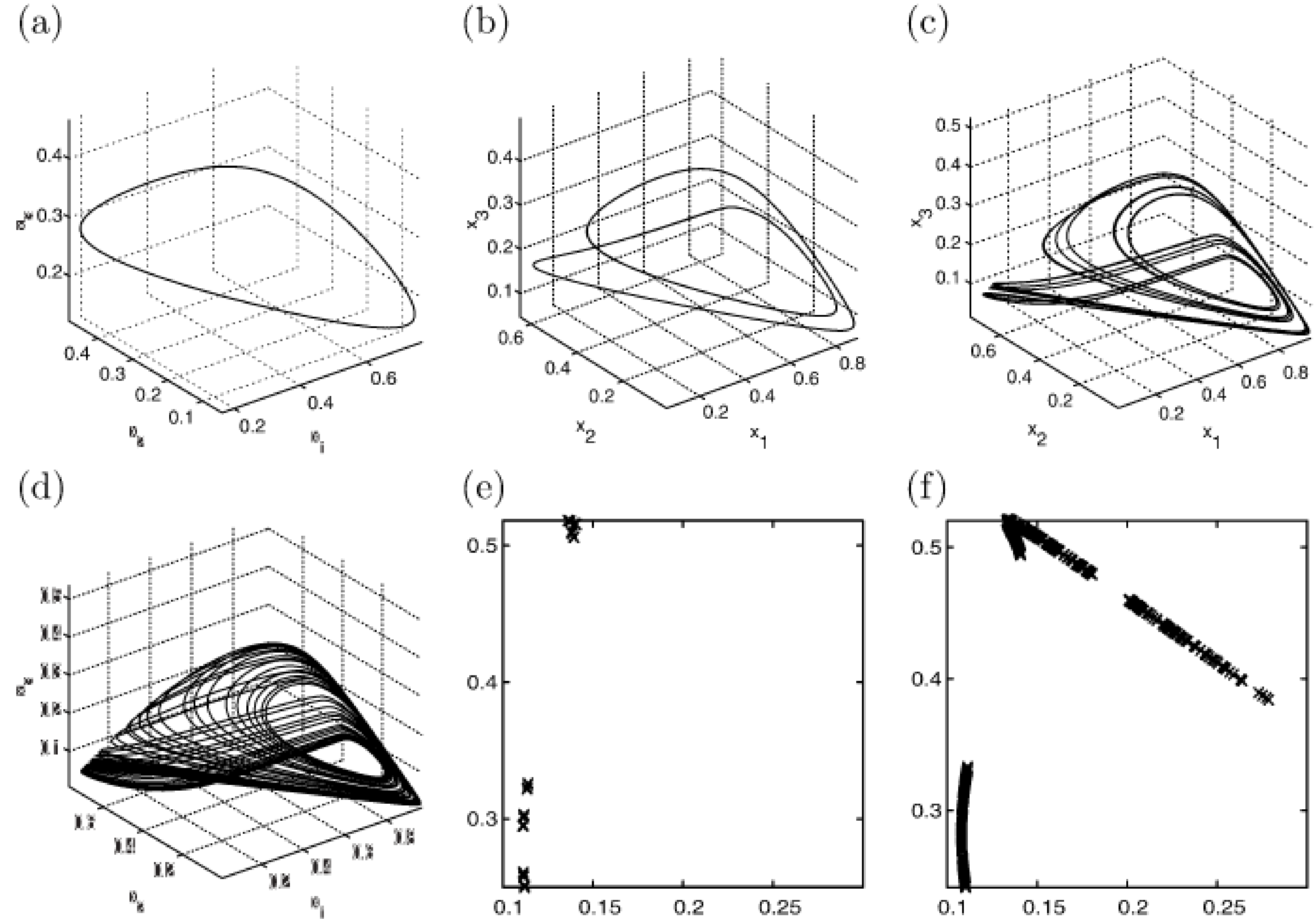


Figure 3.21. Period-doubling bifurcation in the 3 component chemostat. $(m_1, m_2, m_3) = (10, 4, 3.5)$, $(a_1, a_2) = (0.08, 0.23)$, and $a_3 = 0.4, 0.3, 0.225, 0.2$ in (a)-(d), respectively. (e,f) Poincaré section to $x_1 = 0.3$ for $a_3 = 0.225$ and $a_3 = 0.2$, respectively. For $a_3 = 0.2$ the time span is $t \in [0, 200]$.

c) Find for $\lambda = 4$ the structure of the set

$$S_\lambda = \{x : T_\lambda^n(x) \in [0,1] \ \forall n \in \mathbb{N}_0\}$$

What kind of dynamics do you expect in S_λ?

3.9. With $z = x + \mathrm{i}y$ und $c = a + \mathrm{i}b$ the discrete dynamical system

$$\pi_{a,b}(x,y) = (x^2 - y^2 + a, 2xy + b)$$

can be written in the complex form $f_c(z) = z^2 + c$. The Mandelbrot set is sketched in Figure 3.9, and is the set of all $c \in \mathbb{C}$ for which the sequence $(z_n)_{n\in\mathbb{N}}$, defined through $z_{n+1} = f_c(z_n)$, $z_0 = 0$, is bounded.

a) Prove that $\pi_{a,b}$ possesses an asymptotically stable fixed point (x_0, y_0) if $|f_c'(z_0)| < 1$. Express the fixed points of $D\pi_{a,b}(x_0, y_0)$ in terms of f'.

b) Find the fixed points of $\pi_{a,b}$ and compute their stability. Show that the set of all c, for which an asymptotically stable fixed point $z_0(c)$ exists is given by the interior of the cardioid

$$c(t) = \tfrac{1}{4} - \tfrac{1}{4}(1 - \mathrm{e}^{\mathrm{i}t})^2, \quad t \in \mathbb{R}.$$

c) Compute the fixed points of $\pi_{a,b}^2 = \pi_{a,b} \circ \pi_{a,b}$. Show that the non-trivial 2-periodic solutions of $\pi_{a,b}$ are asymptotically stable for $|c + 1| < \frac{1}{4}$.

d) Which periods can be expected in the different parts of the Mandelbrot set? For this consider the eigenvalues of the linearization in the fixed point $z_0(c)$ for different values of c at the boundary of the cardioid.

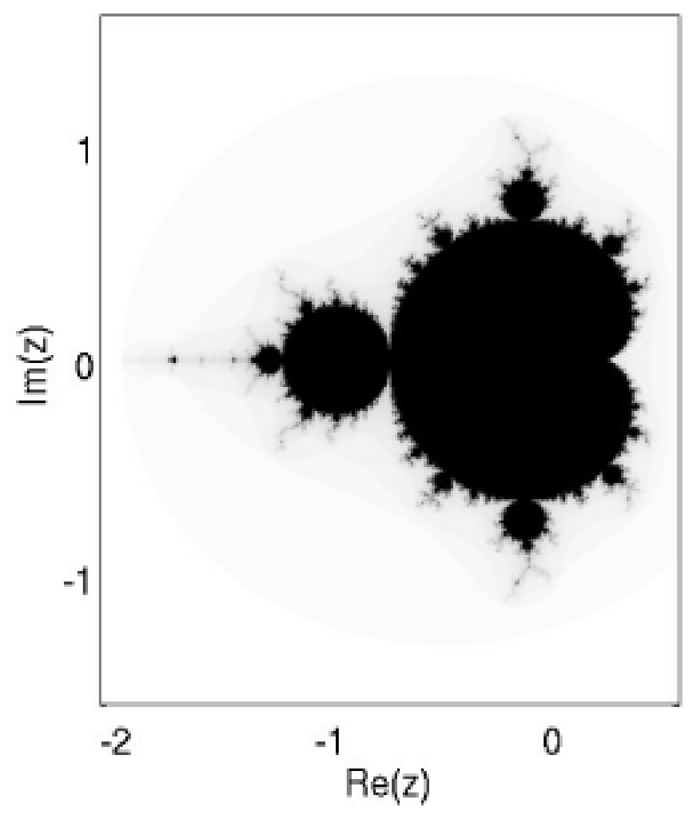


Figure 3.22. The Mandelbrot set, cf. [**Man91**].

Chapter 4

Hamiltonian dynamics

In the systems considered in Chapter 3, the evolution changes the volume of sets in phase space. However, many systems in nature conserve this volume, especially those of classical mechanics. For these systems we will discuss their dynamical properties, as stability and instability, and the occurrence of chaotic behavior. Our starting point of the bifurcation analysis of dissipative systems usually was a system with a globally attracting fixed point. In conservative systems such things cannot exist, and so we start from a completely integrable system, i.e., from a system in which all solutions can in principle be computed explicitly. It is the main purpose of this section to contrast the behavior of dissipative and conservative systems. Hence, essential parts in usual courses about Hamiltonian systems will be skipped. For an overview we refer to the textbooks [**Arn78, Thi88, MH92, HZ11**], or the selection of reprints [**MM87**].

4.1. Basic properties

The basic rule of classical mechanics is that the force $f : \mathbb{R}^d \to \mathbb{R}^d$ acting on a mass point at a position $q \in \mathbb{R}^d$ equals the product of mass m and acceleration $\ddot{q}$, i.e.,

$$m\ddot{q} = f(q). \tag{4.1}$$

For simplicity we set $m = 1$ in the following. In a conservative system, to a given force $f : \mathbb{R}^d \to \mathbb{R}^d$ there exists a potential $U : \mathbb{R}^d \to \mathbb{R}$ with $f = -\partial_q U$. Introducing the momentum variable $p = \dot{q} \in \mathbb{R}^d$ gives the first order system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -\partial_q U \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\partial_p(\|p\|^2) \\ -\partial_q U \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = J\nabla H, \tag{4.2}$$

where $H : \mathbb{R}^{2d} \to \mathbb{R}$ given by $H = \frac{1}{2}\|p\|^2 + U$ is called the Hamiltonian and where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathbb{R}^{2d\times 2d}$$

is a skew symmetric operator with I the identity matrix in $\mathbb{R}^{d\times d}$. All classical mechanical systems can be written as a Hamiltonian system

$$\dot{u} = J\nabla H(u), \tag{4.3}$$

with $u(t) \in \mathbb{R}^{2d}$, $H : \mathbb{R}^{2d} \to \mathbb{R}$ the Hamiltonian, and $J \in \mathbb{R}^{2d\times 2d}$ the skew symmetric operator from above. For ∇H locally Lipschitz-continuous, there exists a unique solution $u = u(t, u_0)$ of (4.3) with initial condition $u|_{t=0} = u_0$, cf. Theorem 2.2.1. An important property of (4.3) is the conservation of energy.

Theorem 4.1.1. *The Hamiltonian H is constant along a solution, i.e., $H(u(t, u_0))$=$H(u_0)$.*

Proof. Let $u = u(t)$ be a solution of the Hamiltonian system (4.3). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}H(u(t)) =(\nabla H)^T\dot{u}(t) = (\nabla H)^T J\nabla H = 0,$$

due to the skew symmetry of J. □

In case $d = 1$ in (4.1) the full phase portrait can be constructed graphically.

Example 4.1.2. Consider an ODE

$$\ddot{x} = f(x) \tag{4.4}$$

with locally Lipschitz-continuous but otherwise arbitrary $f : \mathbb{R} \to \mathbb{R}$. In Figure 4.1 we explain how to draw the phase portrait without any formulas or calculations. Given f, without formulas, we may draw the potential energy $-F(x)$. In the top left we do this by first labeling the 4 zeros of f by $x_1, \ldots, x_4$. Setting, e.g., $-F(x) = -\int_0^x f(\xi)\,\mathrm{d}\xi$ we may plot $-F$ and find that $x_1, \ldots, x_4$ are stationary points of $-F$; here they are extrema since the roots of f are simple. Note that given $-F$ we obtain $E = \frac{1}{2}\dot{x}^2 - F$ by simply adding a parabola in $\dot{x}$ to $-F(u)$ at each x, see the bottom right for an illustration. Thus, minima of $-F$ are minima of E, while maxima of $-F$ are saddle points for E with stable direction $(0, 1)$ and unstable direction $(1, 0)$. To draw orbits we may think of small balls rolling around on the energy surface. For instance, consider a ball starting at $(x, \dot{x}) = (x_5, 0)$. It will slowly start to move to the right, thereby loosing potential energy $-F$ and taking up speed, hence gaining kinetic energy $\frac{1}{2}\dot{x}^2$. At $x = x_2$ it will have maximum speed, and then has to roll uphill, thus loosing kinetic energy but gaining potential energy. Thus, it will roll precisely until x_6, defined by $-F(x_6) = -F(x_5)$, where all kinetic energy has been transformed to potential energy again. The ball will now roll back, and all together we obtain a periodic orbit γ_1. Similar periodic orbits are obtained for all starting positions $(x, 0)$ with $x_2 < x < x_3$, or, equivalently for all $(x, \dot{x})$ inside the region bounded by the homoclinic orbit γ_2 to $(x_3, 0)$ and passing through $(x_7, 0)$. In a similar way all orbits can be constructed graphically. For instance, the orbit γ_3 corresponds to a ball coming from the far left with some large positive speed and rolling through the potential all the way to a position x_8 on the (far) right where it reaches some maximal potential energy $-F(x_8)$ and then rolls back. ⌋

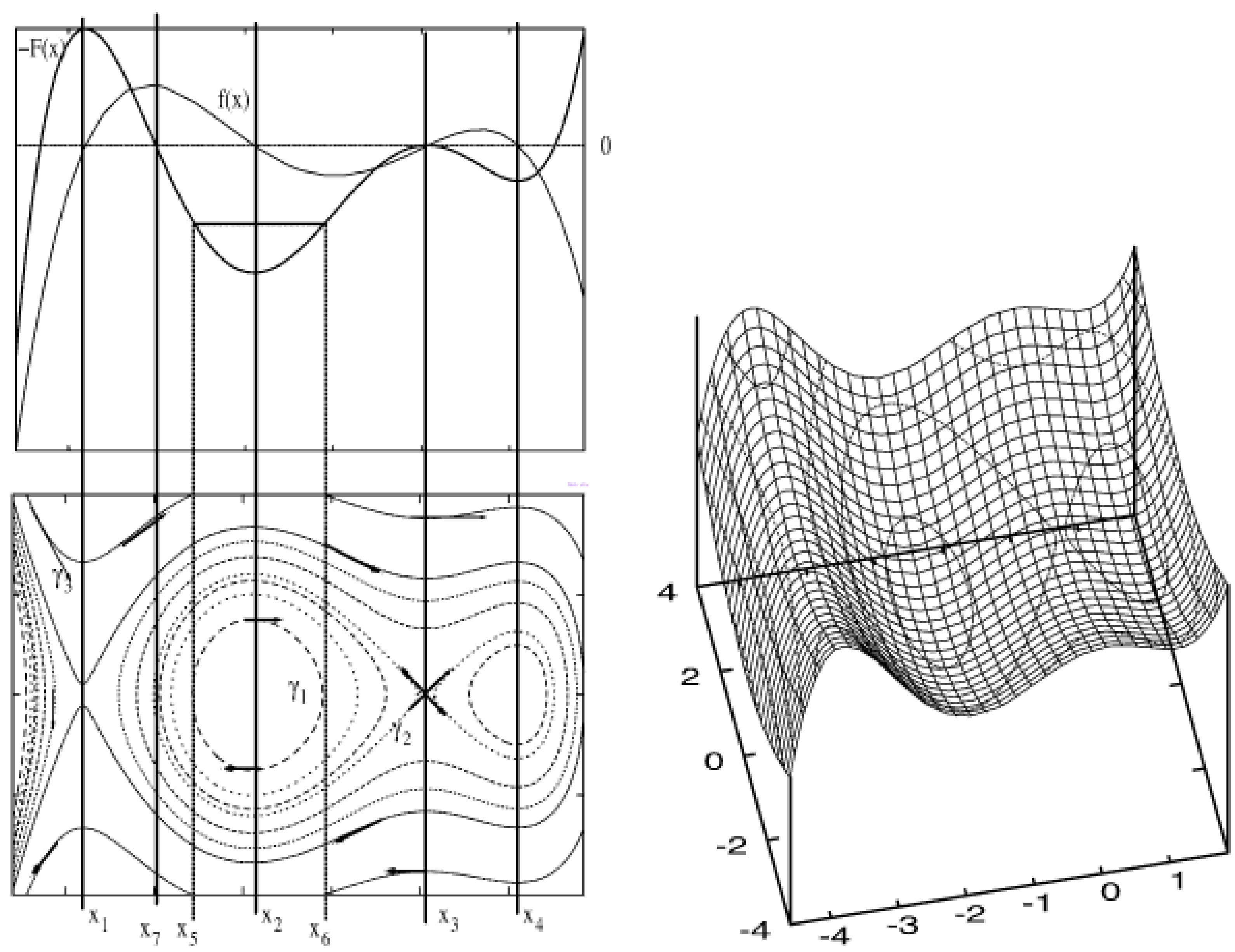


Figure 4.1. Phase portrait for a general scalar Newtonian system $\ddot{x} = f(x)$.

The volume of sets in the phase space of Hamiltonian systems is preserved.

Theorem 4.1.3. (Liouville) *Let*

$$u(t, \Omega) = \{v \in \mathbb{R}^{2d} : \exists u_0 \in \Omega : u(t, u_0) = v\}$$

be the image of a bounded and measurable $\Omega \subset \mathbb{R}^{2d}$ under the map $u(t, \cdot)$, and let μ be the Lebesgue measure in phase space, i.e., $d\mu = 1du$. For all $t \in \mathbb{R}$ we then have

$$\int_{\Omega} \mathrm{d}\mu(u) = \int_{u(t,\Omega)} \mathrm{d}\mu(u).$$

Proof. A change of coordinates yields

$$\int_{u(t,\Omega)} d\mu(u) = \int_\Omega |D_{u_0}u(t,u_0)|\,\mathrm{d}\mu(u_0).$$

The Jacobi matrix $Y(t) = D_{u_0}u(t,u_0)$ satisfies the differential equation

$$\frac{d}{dt}Y(t) = ((D(J\nabla H))(u(t,u_0)))Y(t).$$

Furthermore, the determinant $|Y(t)|$ satisfies

$$\frac{d}{dt}|Y(t)| = \mathrm{trace}(D(J\nabla H))|Y(t)|.$$

See Exercise 4.1 for a proof of this formula in $\mathbb{R}^2$. Using $Y(0) = I$, and

$$\mathrm{trace}(DJ\nabla H) = \sum_{i=1}^{2d} \delta_{ik} \sum_{k=1}^{2d} \partial_{u_k} \sum_{j=1}^{2d} J_{ij}\partial_{u_j} H = \sum_{i=1}^{2d}\sum_{j=1}^{2d} J_{ij}\partial_{u_i}\partial_{u_j} H = 0$$

due to the skew symmetry of J and due to the symmetry of the matrix $(\partial_{u_i}\partial_{u_j}H)_{i,j}$, we obtain the assertion. □

The invariant Lebesgue measure in phase space is called Liouville measure. The theory of measure preserving dynamical systems is the subject of ergodic theory, see, e.g., [**Wal82, Kre85**]. Complicated dynamical behaviour in Hamiltonian systems is described statistically in this theory.

4.1.1. Dynamics near a fixed point. As a direct consequence of the invariance of the phase space volume Hamiltonian systems cannot possess asymptotically stable fixed points. We start with the discussion of the linearization of the system at the fixed point. The Hamiltonian H must be quadratic in order to obtain a linear differential equation, i.e.,

$$H(u) = \frac{1}{2}\langle Mu, u\rangle = \frac{1}{2}\sum_{i,j=1}^{2d} m_{ij}u_iu_j.$$

Then

$$\partial_{u_k} H = \frac{1}{2}\sum_{i,j=1}^{2d}(m_{ij}\delta_{ik}u_j + m_{ij}u_i\delta_{jk}) = \frac{1}{2}\sum_{i=1}^{2d}(m_{ik}+m_{ki})u_i,$$

i.e., w.l.o.g. M can be considered as symmetric. Hence, a linear Hamiltonian system is of the form

$$\dot{u} = JMu$$

with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $M = M^T$.

Lemma 4.1.4. *Let λ be an eigenvalue of JM. Then also $-\lambda$, $\overline{\lambda}$ and $-\overline{\lambda}$ are eigenvalues of JM.*

Proof. We have $\det J = 1$, $J^2 = -I$ and $J^T = -J$. For the characteristic polynomial of JM we obtain

$$\begin{aligned} p(\lambda) &= \det(JM - \lambda I) = \det(J)\det(JM - \lambda I) = \det(J^2 M - \lambda J) \\ &= \det(-M - \lambda J) = \det(-M - \lambda J)\det(J) \\ &= \det(-MJ + \lambda I) = \det((-MJ + \lambda I)^T) \\ &= \det(-J^T M^T + \lambda I) = \det(JM + \lambda I) = p(-\lambda). \end{aligned}$$

Hence, with λ also $-\lambda$ is an eigenvalue. Since JM is a real-valued matrix, the complex conjugate of an eigenvalue must be an eigenvalue, too. ☐

A direct consequence is the following lemma, which again implies that in Hamiltonian systems no asymptotically stable fixed points can exist.

Lemma 4.1.5. *A fixed point of a Hamiltonian system can only be stable if all eigenvalues of the linearization lie on the imaginary axis with same geometric and algebraic multiplicity.*

For general ODEs in case that all eigenvalues lie on the imaginary axis the nonlinear terms decide about stability. For Hamiltonian systems the quadratic approximation of the Hamiltonian at the fixed point gives additional information.

Theorem 4.1.6. *Let $H(u) = \frac{1}{2}u^T Au + \mathcal{O}(\|u\|^3)$ with A strictly positive (or strictly negative) definite. Then $u = 0$ is stable.*

Proof. Let

$$\rho_0(r) = \min\{H(u) : |u| = r\}, \quad \rho_1(r) = \max\{H(u) : |u| = r\}.$$

Then

$$\rho_0(r) = \frac{1}{2}\lambda_{\min} r^2 + \mathcal{O}(r^3), \quad \rho_1(r) = \frac{1}{2}\lambda_{\max} r^2 + \mathcal{O}(r^3),$$

where $\lambda_{\min} > 0$, respectively $\lambda_{\max} > 0$, is the smallest, respectively the largest eigenvalue, of the positive definite matrix A. Then there exists an $r_0 > 0$, such that

$$\rho_0(r) \geq \frac{1}{4}\lambda_{\min} r^2 \quad \text{and} \quad \rho_1(r) \leq \lambda_{\max} r^2.$$

for all $r \in [0, r_0]$. Given $\varepsilon > 0$ we choose

$$0 \leq \delta \leq \sqrt{\frac{\lambda_{\min}}{4\lambda_{\max}}} \min(\varepsilon, r_0).$$

Then

$$\begin{aligned} |u(t,u_0)|^2 &\leq \frac{4}{\lambda_{\min}} \rho_0(u(t,u_0)) \leq \frac{4}{\lambda_{\min}} H(u(t,u_0)) \\ &= \frac{4}{\lambda_{\min}} H(u_0) \leq \frac{4}{\lambda_{\min}} \rho_1(u_0) \leq \frac{4\lambda_{\max}}{\lambda_{\min}} |u_0|^2 \leq \frac{4\lambda_{\max}}{\lambda_{\min}} \delta^2 \leq \varepsilon^2. \end{aligned}$$

Hence, the solution $u = u(t, u_0)$ cannot leave the ε-neighborhood, if the initial condition u_0 is contained in the δ-neighborhood. $\square$

Example 4.1.7. Let $H = q^2 + p^2$. Then

$$\dot{q} = p, \quad \dot{p} = -q.$$

The orbits are circles with radius $\sqrt{H}$ and $u = (q, p) = (0, 0)$ is stable. $\rfloor$

4.1.2. Lyapunov's subcenter theorem. Our next goal is the existence of periodic solutions. It turns that in each neighborhood of a fixed point with imaginary eigenvalues we can always find periodic solutions if the eigenvalues satisfy some non-resonance condition. Before we prove this result, we need two preparations.

First we introduce angle and action variables for linear systems. We consider the harmonic oscillator

$$\ddot{q} = -\omega^2 q,$$

which we write as Hamiltonian system

$$\dot{q} = \omega p = \partial_p H, \qquad \dot{p} = -\omega q = -\partial_q H,$$

with Hamiltonian $H = \frac{1}{2}\omega(p^2 + q^2)$. Introducing polar coordinates

$$q = \sqrt{2I}\cos(\phi), \qquad p = \sqrt{2I}\sin(\phi)$$

shows that the new variables ϕ and I satisfy the Hamiltonian system

$$\dot{\phi} = \omega = \partial_I H, \qquad \dot{I} = 0 = -\partial_\phi H$$

with Hamiltonian $H = \omega I$. The 2π-periodic variable ϕ is called angle variable and the variable I which is preserved under the flow is called action variable. Such variables play a fundamental role in the description of completely integrable Hamiltonian systems in the following.

Secondly we explain a reduction method for systems which are at least partly given in action angle variables. Let $H = H(\phi, q, I, p)$ be 2π-periodic w.r.t. ϕ. Moreover, assume that $\partial_I H > 0$ in some open subset of $\mathbb{R}^{2d}$. Then $H(\phi, q, I, p) = h$ can be solved for $I = -K(q, p, \phi, h)$.

From $\frac{\mathrm{d}}{\mathrm{d}t}H = 0$ and $H(\phi, q, -K(q, p, \phi, h), p) = h$ it follows that

$$\begin{aligned}
\partial_{q_i} H + \partial_I H \cdot (-\partial_{q_i} K) =& 0,\\
\partial_{p_i} H + \partial_I H \cdot (-\partial_{p_i} K) =& 0,
\end{aligned}$$

and hence using $\frac{\mathrm{d}}{\mathrm{d}t}\phi = \partial_I H$ and $\frac{\mathrm{d}}{\mathrm{d}t}\phi = (\frac{\mathrm{d}}{\mathrm{d}\phi}t)^{-1}$ yields

$$\partial_\phi q_i = \partial_t q_i \partial_\phi t = \partial_{p_i} H / \partial_I H = \partial_{p_i} K(q, p, \phi, h), \tag{4.5}$$

$$\partial_\phi p_i = -\partial_t p_i \partial_\phi t = \partial_{q_i} H / \partial_I H = -\partial_{q_i} K(q, p, \phi, h). \tag{4.6}$$

This is a $2(d-1)$-dimensional Hamiltonian system which is 2π-periodic w.r.t. the new time variable ϕ.

With these two preparations we are now going to prove

Theorem 4.1.8. (Lyapunovs subcenter theorem) *Let $u = 0$ be a fixed point of the Hamiltonian system $\dot{u} = J\nabla H(u)$. Let $\pm i\omega$, with $\omega \neq 0$, be simple eigenvalues of the linearization $JD^2H(0)$, and let all other eigenvalues λ_j fulfill $\lambda_j \neq in\omega$ for all $n \in \mathbb{Z}$. Then there is a neighborhood U of 0 and a two-dimensional manifold $M \subset U$ which is filled with periodic solutions with period close to $2\pi/\omega$. Moreover, M is tangential to the subspace spanned by the eigenvectors which are associated to the eigenvalues $\pm i\omega$.*

Proof. We seek small solutions $x = \varepsilon\widetilde{x}$, with $0 < \varepsilon \ll 1$ a small parameter, and consider the rescaled Hamiltonian

$$\widetilde{H}_\varepsilon(\widetilde{x}) = \frac{1}{\varepsilon^2}H(\varepsilon\widetilde{x}).$$

By this rescaling the quadratic part of the Hamiltonian stays independent of ε, whereas the higher order terms become small. W.l.o.g. assume that

$$D^2\widetilde{H}_\varepsilon(0) = D^2H(0) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & A \end{pmatrix},$$

with $A \in \mathbb{R}^{2(d-1)\times 2(d-1)}$. This form can always be achieved by interchanging the second and the $(d+1)$th coordinate. The coordinates associated to the matrix A are denoted by $\widehat{x}$. For the first two coordinates $\widetilde{q}_1$ and $\widetilde{p}_1$ we introduce polar coordinates $\widetilde{q}_1 = \sqrt{2I}\cos(\phi)$ and $\widetilde{p}_1 = \sqrt{2I}\sin(\phi)$. In the new coordinates we have

$$H_\varepsilon(I, \phi, \widehat{x}) = \omega I + \frac{1}{2}\widehat{x}^T A\widehat{x} + \mathcal{O}(\varepsilon).$$

We look for solutions with $H_\varepsilon(I, \phi, \widehat{x}) = \omega$. For $\omega \neq 0$ we have $\partial_I H_\varepsilon \neq 0$ in a neighborhood of the origin such that $H_\varepsilon(I, \phi, \widehat{x}) = \omega$ can be solved w.r.t. I. We write this solution as $I = 1 - K_\varepsilon(\phi, \widehat{x})$. Inserting this ansatz in $H_\varepsilon(I, \phi, \widehat{x}) = \omega$ yields

$$K_\varepsilon(\phi, \widehat{x}) = \frac{1}{2\omega}\widehat{x}^T A\widehat{x} + \mathcal{O}(\varepsilon).$$

By the above reduction we obtain a 2π-periodic Hamiltonian system with new time variable ϕ and associated Poincaré map $\psi_\varepsilon(\widehat{x}_0) = \widehat{x}(2\pi, \widehat{x}_0)$. A fixed point $\widehat{x}^*$ of the Poincaré map ψ_ε yields a periodic solution $\widehat{x}(t, \widehat{x}^*) = \widehat{x}(t + 2\pi, \widehat{x}^*)$. Thus, a periodic solution can be obtained via a zero of the function $F(\widehat{x}^*, \varepsilon) = \psi_\varepsilon(\widehat{x}^*) - \widehat{x}^*$. We have $F(0,0) = 0$ since $K_0(\phi, \widehat{x}) = \frac{1}{2\omega}\widehat{x}^T A\widehat{x}$ and thus $\widehat{x} = 0$ is a solution of the associated linear autonomous Hamiltonian system $\dot{\widehat{x}} = \frac{1}{\omega}JA\widehat{x}$. For the same reason we have $D_{\widehat{x}^*}F(0,0) =$

$e^{\frac{2\pi}{\omega}JA} - I$. Due to the non-resonance assumption, $D_{\widehat{x}^*}F(0,0)$ has only non-zero eigenvalues and hence is invertible. Then, by the implicit function theorem, $F(\widehat{x}^*, \varepsilon) = 0$ can be solved for $\widehat{x}^* = \widehat{x}^*(\varepsilon)$. The associated solution $(I, \phi, \widehat{x})$ is non-trivial since $I \neq 0$. The period is

$$\int_0^T \mathrm{d}t = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\dot{\phi}} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\partial_I H} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\omega + \mathcal{O}(\varepsilon)} = \frac{2\pi}{\omega} + \mathcal{O}(\varepsilon).$$

This family of periodic solutions is tangential to $\operatorname{span}\{I, \phi\}$ since $\|\widehat{x}\| = \mathcal{O}(\varepsilon)$. □

4.2. Some celestial mechanics

To give some illustrations of computations in Hamiltonian dynamics we review some very basic celestial mechanics. A good reference is [**Gut94**].

4.2.1. The 1-body problem. Let $q \in \mathbb{R}^3$ be the position of a mass point, e.g., earth, that moves in a radially symmetric potential $U(q) = \widetilde{U}(\|q\|)$, i.e., $U(q)$ only depends on $\|q\|$, e.g. the gravitational potential of the sun. Then

$$\ddot{q} = -\nabla U(q) = -\widetilde{U}'(\|q\|)q/\|q\|,$$

or, in Hamiltonian form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = J\nabla H(q,p), \quad \text{with} \quad H(q,p) = U(q) + \frac{1}{2}\|p\|^2.$$

This is a 6-dimensional first order ODE, or, more precisely a Hamiltonian system with 3 degrees of freedom. Using the angular momentum, see Exercise 4.8,

$$C = q \times \dot{q}, \quad a \times b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

the dimension can be reduced. Let $C \neq 0$. Then C is orthogonal to the orbital plane $\{\alpha q + \beta\dot{q} : \alpha, \beta \in \mathbb{R}\}$. The area swept until t then is by Leibniz's sector formula

$$F(t) = \frac{1}{2}\int_0^t |q(s) \times \dot{q}(s)|\,\mathrm{d}s.$$

This yields **Kepler's second law**: the line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time, or in modern formulation: $\frac{\mathrm{d}}{\mathrm{d}t}F(t) = \frac{1}{2}|q(t) \times \dot{q}(t)| = \frac{1}{2}C$.

Now assume that the orbital plane is the q_1-q_2 plane, let $q = (q_1, q_2) = (x, y)$, and introduce polar coordinates, i.e.

$$q = \begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

Then

$$p := \dot{q} = \dot{r}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix} + r\dot{\phi}\begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix},$$

and C and H in polar coordinates become

$$C = r^2\dot{\phi} \neq 0, \quad H = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \widetilde{U}(r). \tag{4.7}$$

From (4.7) we obtain $\dot{\phi} \neq 0$ and $\dot{r}^2 = 2H - r^2\dot{\phi}^2 - 2\widetilde{U}(r)$. This yields a scalar first order equation as follows. Locally we can assume $r = r(\phi)$; then

$$\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}r(\phi) = (\frac{\mathrm{d}}{\mathrm{d}\phi}r)\dot{\phi} \Rightarrow r' := \frac{\mathrm{d}}{\mathrm{d}\phi}r = \frac{\dot{r}}{\dot{\phi}}.$$

In particular

$$(r')^2 = \frac{2(H-U)}{\dot{\phi}^2} - r^2 = \frac{2(H-U)r^4}{C^2} - r^2,$$

which yields the Clairaut ODE

$$r' = \pm g(r), \quad \text{with} \quad g(r) = \sqrt{2C^{-2}(H - U(r))r^4 - r^2}.$$

Instead of r we use the inverse radius $\sigma{=}1/r$, which yields the so called **fundamental equation of the 1-body problem**

$$\sigma' = g(1/\sigma)\sigma^2.$$

For the gravitational potential $U(r) = -Ar^{-1}$ we obtain

$$\sigma' = -\sqrt{-\sigma^2 + \alpha\sigma + \beta}, \quad \text{with} \quad \alpha = 2A/C^2, \quad \beta = 2H/C^2. \tag{4.8}$$

Lemma 4.2.1. *a) We have $H \geq -A^2/(2C^2)$ (lower energy bound).*
b) For $H = -A^2/(2C^2)$ the orbit is a circle with radius C^2/A.

Proof. Completing the square we write $-(\sigma')^2 = \sigma^2 - \alpha\sigma - \beta = (\sigma - \frac{\alpha}{2})^2 - \delta/4$, where $\delta = \alpha^2 + 4\beta$. Thus $\delta \geq 0$ and hence $H \geq -A^2/(2C)^2$. For $H = -A^2/(2C)^2$ we have $\delta = 0$ and hence $\sigma' = 0$ and $\sigma \equiv \alpha/2$. □

Henceforth let $\delta = \alpha^2 + 4\beta > 0$. We seek solutions of

$$\sigma' = -g(\sigma) \tag{4.9}$$

with $g(\sigma) = \sqrt{-\sigma^2 + \alpha\sigma + \beta}$ and $\sigma \in I := [\frac{\alpha - \sqrt{\delta}}{2}, \frac{\alpha + \sqrt{\delta}}{2}]$ to have a real radicand. In particular, an unbounded orbit is only possible for

$$0 \in I \Leftrightarrow \alpha \leq \sqrt{\delta} \Leftrightarrow \beta \geq 0 \Leftrightarrow H \geq 0,$$

while the orbit is always bounded if $H < 0$. Moreover, from $\sigma \leq \frac{\alpha+\sqrt{\delta}}{2}$ we obtain a minimal distance of the orbit to the origin, i.e.,

$$r \geq r_{\min} := \frac{2}{\alpha + \sqrt{\delta}}.$$

Let $\sigma(0) = \sigma_0$ with $\max\{0, \frac{\alpha-\sqrt{\delta}}{2}\} < \sigma_0 < \frac{\alpha+\sqrt{\delta}}{2}$. Then we have a local solution as g is locally Lipschitz near σ_0. Substituting $u = \frac{1}{2}\sqrt{\delta}x + \frac{\alpha}{2}$ we obtain

$$\begin{aligned}\phi &= \int_{\sigma_0}^{\sigma} \frac{\mathrm{d}u}{-g(u)} = \int_{\sigma_0}^{\sigma} \frac{\mathrm{d}u}{-\sqrt{-u^2+\alpha u+\beta}} = \int_{\widehat{\sigma}_0}^{\widehat{\sigma}} \frac{\mathrm{d}x}{-\sqrt{1-x^2}} \\ &= \arccos\widehat{\sigma} - \arccos\widehat{\sigma}_0,\end{aligned}$$

with $\widehat{\sigma}_0 = \frac{2}{\sqrt{\delta}}(\sigma_0 - \frac{\alpha}{2})$ and $\widehat{\sigma} = \frac{2}{\sqrt{\delta}}(\sigma - \frac{\alpha}{2})$. From $-1 < \widehat{\sigma}_0 < 1$ we obtain $0 < \arccos(\widehat{\sigma}_0) < \pi$ and thus

$$\frac{2}{\sqrt{\sigma}}(\sigma - \alpha/2) = \widehat{\sigma} = \cos(\phi + b) \quad \text{with} \quad b = \arccos(\widehat{\sigma}_0). \tag{4.10}$$

W.l.o.g. we choose the initial condition $\sigma_0 = 1/r_{\min}$ and obtain

$$r(\phi) = \frac{p}{1 + e\cos(\phi)} \quad \text{with} \quad p = C^2/A, \quad e = \sqrt{1 + 2HC^2/A^2}. \tag{4.11}$$

We distinguish three cases.

(1) $0 \le e < 1 \Leftrightarrow H < 0$: Then $r(\phi)$ is defined for all $\phi \in \mathbb{R}$ and 2π-periodic in ϕ. Going back to cartesian coordinates we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix},$$

and $\cos\phi = (p/r - 1)/e$ yields $(p - ex)^2 = r^2 = x^2 + y^2$, and thus

$$\left(x + \frac{ep}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{p^2}{(1-e^2)^2}.$$

This is **Kepler's first law:** The body moves on an ellipse with focal points $(0,0)$ and $(-(2ep)/(1-e^2), 0)$, numerical eccentricity e and major semi-axis $a = -A/(2H)$. The point $(r_{\min}, 0)$ is called *peri center* (*perihel* for a planet in the solar system) and $(r_{\max}, \pi)$ is called *apo center* (*apohel*). Examples for numerical eccentricities e are $e = 0.0167$ for Earth, $e = 0.2056$ for Mercury, and $e = 0.9673$ for Halley's comet. The relatively large eccentricity of Mercury is of great importance historically since already in the 19th century it allowed the observation of the perihel precession of Mercury: after each elliptical orbit Mercury's perihel is shifted by a few angular seconds. This contradicts the above (newtonian) calculations, but could be explained by Einstein's relativity theory.

(2) $e = 1 \Leftrightarrow H = 0$: the existence interval is $-\pi < \phi < \pi$, and geometrically the orbit is a parabola opening to the left, $y^2 = -px + p^2$.

(3) $e > 1 \Leftrightarrow H > 0$: the orbit is the hyperbola $y^2 = (e^2-1)x^2 - 2epx + p^2$.

Thus we found the orbits for the 1-body problem in implicit form and without time dependence, determined by parameters H, A and C. Next, the orbits can be characterized via initial conditions and the time-dependence

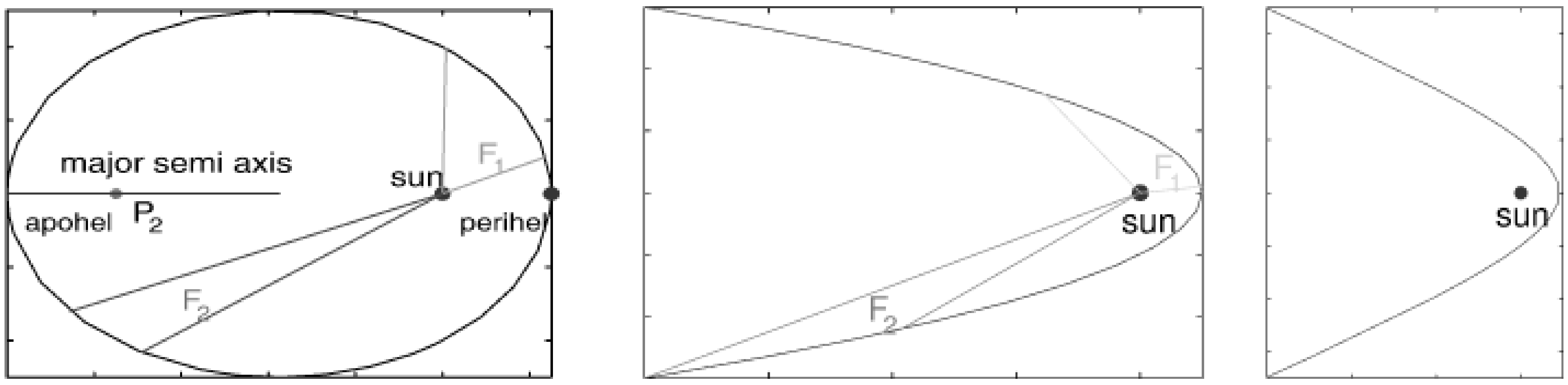


Figure 4.2. Kepler-ellipse, parabola and hyperbola. The areas F_1, F_2 are meant to illustrate Kepler's 2nd law.

can be reintroduced. This allows to derive **Kepler's 3rd law:** if a_γ is the length of an orbit's major semi-axis, and T_γ its period, then T_γ^2/a_γ^3 is independent of the orbit γ. More precisely, in this calculus we have $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$ for all orbits in the solar systems, where G is the gravitational constant and M the mass of the sun, which is a very good approximation to observations. This is only natural, as Kepler derived his laws from observations.

4.2.2. The restricted 3-body problem. N bodies which move under the influence of gravity have the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i\neq j} \frac{m_i m_j}{|q_i - q_j|}.$$

The solution of the associated differential equations, and the associated question about the mechanical stability of our solar system, have been considered as essential for mankind. However, it turned out that only the two body problem ($N = 2$), see above and Exercise 4.9, can be solved explicitly and already the three body problem shows chaotic behavior.

There is one intermediate problem, namely the so called restricted three body problem. There it is assumed that the third body K_3 has a very small mass compared to the other two bodies K_1 and K_2. The restricted three body problem is obtained by neglecting the forces of K_3 on K_1 and K_2, such that their motion is not affected by K_3, i.e., they move on Kepler ellipses around their center of mass.

In a coordinate system which moves with the two larger bodies of reduced masses $\mu = \frac{m_1}{m_1+m_2}$ and $1-\mu$, which lie fixed in $-\mu$ and $1-\mu$, the Hamiltonian for the motion of the third body is given by

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_2 p_1 - q_1 p_2 + U(q),$$

where

$$U(q) = -\frac{q_1^2}{2} - \frac{q_2^2}{2} - \frac{1-\mu}{\sqrt{(q_1+\mu)^2 + q_2^2 + q_3^2}} - \frac{\mu}{\sqrt{(q_1 - 1 + \mu)^2 + q_2^2 + q_3^2}}.$$

The second and third term in H and the first term in U come from the Coriolis force in the rotating coordinate system. There are five equilibria, called Lagrangian points, shown in Figure 4.3.

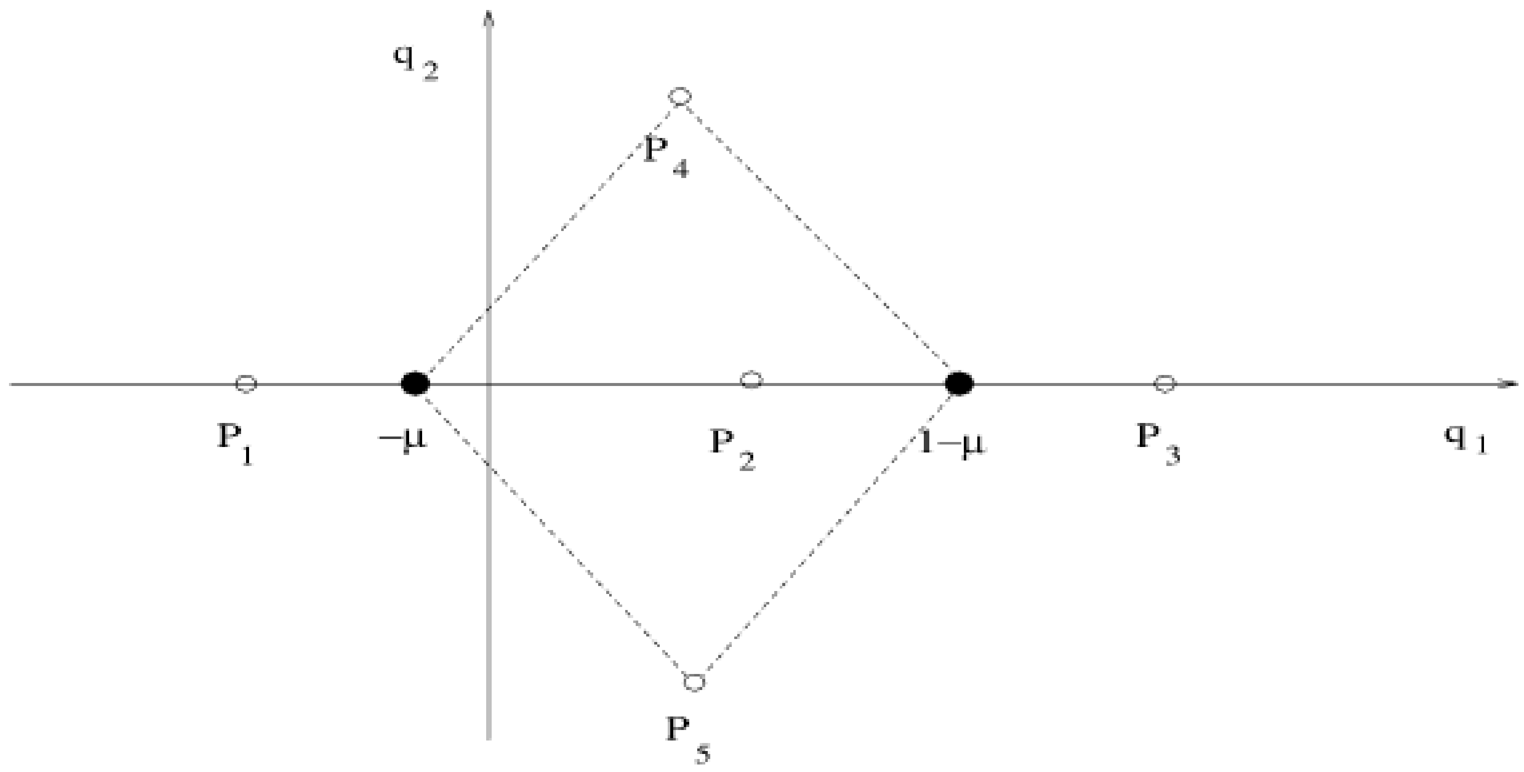


Figure 4.3. The equilibria in the restricted three body problem.

For the stability of these equilibria we first consider

$$M = D^2H(u) = \begin{pmatrix} \partial_{q_1}^2 U & \partial_{q_1}\partial_{q_2}U & 0 & 0 & -1 & 0 \\ \partial_{q_1}\partial_{q_2}U & \partial_{q_2}^2 U & 0 & 1 & 0 & 0 \\ 0 & 0 & \partial_{q_3}^2 U & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the linearisation JM in these points (especially $\partial_{q_j}\partial_{q_3}U|_{P_i} = 0$ for $j = 1, 2$ and $i = 1, \ldots, 5$) we thus obtain

$$JM = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\partial_{q_1}^2 U & -\partial_{q_1}\partial_{q_2}U & 0 & 0 & 1 & 0 \\ -\partial_{q_1}\partial_{q_2}U & -\partial_{q_2}^2 U & 0 & -1 & 0 & 0 \\ 0 & 0 & -\partial_{q_3}^2 U & 0 & 0 & 0 \end{pmatrix}.$$

We find that the q_3, p_3-part decouples and leads to the eigenvalue problem

$$0 = \lambda^2 + \partial_{q_3}^2 U.$$

Since $\partial_{q_3}^2 U > 0$ we have $\lambda_{1,2} \in \mathrm{i}\mathbb{R}$. For the remaining eigenvalues we have

$$0 = \lambda^4 + \lambda^2(\partial_{q_1}^2 U + \partial_{q_2}^2 U + 4) + (\partial_{q_1}^2 U)(\partial_{q_2}^2 U) - (\partial_{q_1}\partial_{q_2}U)^2.$$

It turns out that the points P_1, P_2 and P_3 are saddles and therefore unstable. In the points P_4 and P_5 we find

$$q_1=\frac{1}{2}-\mu,\ q_2=\pm\frac{\sqrt{3}}{2},\ \partial^2_{q_1}U=-\frac{3}{4},\ \partial^2_{q_2}U=-\frac{9}{4},\ (\partial_{q_1}\partial_{q_2}U)^2=-\frac{3\sqrt{3}}{2}(\frac{1}{2}-\mu).$$

For $4(\frac{1}{2}-\mu)^2<1$ the eigenvalues are purely imaginary, i.e., the points P_4 and P_5 are linearly stable. Unfortunately M is indefinite, such that we cannot conclude on the nonlinear stability of P_4 and P_5 with the above theorem. Nevertheless P_4 and P_5 are realized in nature and play an important role for space missions. For instance, Sun and Jupiter can be taken as the big bodies, and in an angle of 60 degrees before and after Jupiter on his orbit there are the so the called Greeks and Trojans, some families of asteroids.

4.3. Completely integrable systems

If there are several non-resonant eigenvalues on the imaginary axis, then there are several families of periodic solutions. In this and the following section we discuss situations with even more complex structures. We start with the linear Hamiltonian system

$$\dot{x}=JMx. \tag{4.12}$$

We assume that all eigenvalues $\mathrm{i}\omega_j$ of the matrix JM are semi-simple and on the imaginary axis. Then the system can be transformed into

$$\dot{q_j}=\omega_j p_j,\quad \dot{p_j}=-\omega_j q_j,\quad j=1,\dots,d, \tag{4.13}$$

i.e., into a Hamiltonian system with Hamiltonian

$$H=\sum_{j=1}^{d}\frac{\omega_j}{2}(q_j^2+p_j^2).$$

Clearly this system is the direct sum of d Hamiltonian systems with d independent Hamiltonians $H_j=\frac{\omega_j}{2}(q_j^2+p_j^2)$. The $I_j=\frac{1}{2\omega_j}H_j$ are conserved also for the flow of (4.12), i.e., $\frac{\mathrm{d}}{\mathrm{d}t}I_j(x(t))=0$ for solutions $x=x(t)$ of (4.12). For the j^{th} system the orbits are circles, i.e.,

$$q_j+ip_j=\sqrt{2I_j}\mathrm{e}^{i\phi_j}\quad\text{with}\quad \phi_j(t)=\phi_j(0)+\omega_j t \bmod 2\pi.$$

For (4.12) the phase space decomposes into d-dimensional tori

$$\{u\in\mathbb{R}^d: I_1=c_1,\dots,I_d=c_d\}.$$

For one or more vanishing I_js we have dimensions of the tori between 1 and d. The d-dimensional tori contain so called quasi-periodic solutions

$$x(t)=g(\omega_1 t,\dots,\omega_d t)$$

with $g : S^1 \times \ldots \times S^1 \to \mathbb{R}^{2d}$. If the non-resonance condition

$$\omega \cdot n = \omega_1 n_1 + \ldots + \omega_d n_d \neq 0 \quad \text{for all} \quad (n_1, \ldots, n_d) \in \mathbb{Z} \times \ldots \times \mathbb{Z} \setminus \{0, \ldots, 0\}$$

holds, then the orbits are dense in the associated d-dimensional torus. If for instance $\omega_1 = 2\omega_2$ and all other ω_i are non-resonant, then the solutions are dense in $d-1$-dimensional tori.

In the following we study whether this situation persists under perturbations or not. We expect that it is more simple to destroy a torus filled with low-dimensional solutions associated to a set of resonant ωs than a torus with dense solutions associated to non-resonant ωs. Therefore, we expect non-resonance conditions to play an important role. Moreover, such tori are not only important for linear Hamiltonian systems but also for special nonlinear systems, which are called completely integrable, see below.

A coordinate transform $y = T(x)$ will in general destroy the Hamiltonian structure of a Hamiltonian system. Only so called symplectic transformations keep the Hamiltonian structure. For $\widetilde{H}(y) = H(x)$ we have

$$\partial_{x_j} H(T(x)) = \sum_{k=1}^{2d} \partial_{y_k} \widetilde{H}(y) \partial_{x_j} y_k,$$

and hence

$$\dot{y} = (DT)\dot{x} = (DT) J \nabla_x H(x) = (DT) J (DT)^T \nabla_y \widetilde{H}(y) = J \nabla_y \widetilde{H}(y),$$

if T is a so called symplectic transformation.

Definition 4.3.1. *Let $J^{-1} = -J = J^T$. The bilinear form*

$$\omega(v_1, v_2) = v_1^T J v_2$$

is called the symplectic structure induced by J. A transformation $y = T(x)$ is called canonical or symplectic if

$$((DT)(x)) J (DT)^T(x) = J \quad \forall x \in \mathbb{R}^{2d}.$$

A Hamiltonian system is called completely integrable if it can be transformed into the form

$$\dot{I}_j = -\partial_{\phi_j} H = 0, \quad \dot{\phi}_j = \partial_{I_j} H = \omega_j, \quad j = 1, \ldots, d \tag{4.14}$$

by a symplectic transformation. The Hamiltonian $H = H(I_1, \ldots, I_d)$ and the frequencies $\omega_j = \omega_j(I_1, \ldots, I_d)$ only depend on the conserved quantities $I_1, \ldots, I_d$. If the set

$$\{x \in \mathbb{R}^{2d} : I_1 = const_1, \ldots, I_d = const_d\}$$

is smooth and compact then it is a d-dimensional torus. The I_j and ϕ_j are called action and angle variables, respectively.

Lemma 4.3.2. *The map $x_0 \mapsto x(t, x_0)$ is symplectic for all t.*

Proof. Consider $R(t,\cdot) = (Dx(t,\cdot))J(Dx(t,\cdot))^T$. Then $R|_{t=0} = J$ and R solves the linear ODE

$$\begin{aligned}\dot{R} =&(D\dot{x})J(Dx)^T + (Dx)J(D\dot{x})^T\\ =&JD^2H(Dx)J(Dx)^T + DxJ(JD^2H(Dx))^T\\ =&JD^2HR + R(JD^2H)^T.\end{aligned}$$

Since $(D^2H)^T = D^2H$ and $J^T = -J$ we have

$$JD^2HJ + J(JD^2H)^T = JD^2HJ + J(D^2H)^T J^T = 0.$$

Hence, $R = J$ is the unique solution. □

Example 4.3.3. Let $F = F(\widetilde{I}, \phi)$. Then the map induced by $\widetilde{\phi} = \partial_{\widetilde{I}} F$ and $I = \partial_\phi F$ is symplectic, cf. [**Arn78**, §48]. ⌋

There are various sufficient conditions that ensure that a Hamiltonian system is completely integrable. Here we will only cite one major criterion.

Definition 4.3.4. *Let $F : \mathbb{R}^{2d} \to \mathbb{R}$ and $G : \mathbb{R}^{2d} \to \mathbb{R}$ be in C^1. Then*

$$\{F, G\} = (\nabla F)^T J \nabla G$$

is called the Poisson bracket of F and G.

Theorem 4.3.5. (Liouville's theorem) *Let $H : \mathbb{R}^{2d} \to \mathbb{R}$, and let $I_1 = H, I_2, \ldots, I_d$ be independent integrals in involution, i.e. $\{I_j, I_k\} = 0$ for $i, j = 1, \ldots, d$. Then the Hamiltonian system is completely integrable.*

Proof. See [**Arn78**, §49]. □

4.4. Perturbations of completely integrable systems

The question occurs how robust completely integrable systems are under perturbations. The answer turns out to be rather delicate and has to do with number theory. The starting situation is as follows. Consider a Hamiltonian of the form

$$H(\phi, I) = H_0(I) + \varepsilon H_1(\phi, I, \varepsilon), \tag{4.15}$$

with $I \in \mathbb{R}^d$, $\phi \in T^d$, and ε a small parameter. The associated Hamiltonian system reads

$$\dot{\phi} = \partial_I H_0 + \varepsilon \partial_I H_1(I, \phi), \qquad \dot{I} = -\varepsilon \partial_\phi H_1(I, \phi).$$

Hence, the action variable I only changes slowly in time. The idea is to find a change of coordinates such that the transformed system is of the original completely integrable form. According to Example 4.3.3 we obtain a symplectic transformation if we take a so called generating function $F : \mathbb{R}^{2d} \to \mathbb{R}$ and define a relation between the old and new variables through

$$\widetilde{\phi} = \partial_{\widetilde{I}} F(\phi, \widetilde{I}) \qquad \text{and} \qquad I = \partial_\phi F(\phi, \widetilde{I}).$$

We look for a transformation near the identity and therefore choose

$$F(\phi, \widetilde{I}) = \phi \widetilde{I} + \varepsilon f(\phi, \widetilde{I}), \tag{4.16}$$

which yields

$$\widetilde{\phi} = \partial_{\widetilde{I}} F = \phi + \varepsilon \partial_{\widetilde{I}} f, \qquad I = \partial_\phi F = \widetilde{I} + \varepsilon \partial_\phi f,$$

and therefore

$$\phi = \widetilde{\phi} - \varepsilon \partial_{\widetilde{I}} f + \mathcal{O}(\varepsilon^2), \qquad I = \widetilde{I} + \varepsilon \partial_{\widetilde{\phi}} f + \mathcal{O}(\varepsilon^2).$$

Plugging this into the Hamiltonian gives

$$\begin{aligned} \widetilde{H}(\widetilde{\phi}, \widetilde{I}) =& H_0(\widetilde{I} + \varepsilon \partial_{\widetilde{\phi}} f) + \varepsilon H_1(\widetilde{\phi}, \widetilde{I}, 0) + \mathcal{O}(\varepsilon^2) \\ =& H_0(\widetilde{I}) + \varepsilon[\partial_{\widetilde{I}} H_0 \cdot \partial_{\widetilde{\phi}} f + H_1(\widetilde{\phi}, \widetilde{I}, 0)] + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The idea is to eliminate the terms of order $\mathcal{O}(\varepsilon)$ by finding f such that

$$\partial_{\widetilde{I}} H_0 \cdot \partial_{\widetilde{\phi}} f + H_1(\widetilde{\phi}, \widetilde{I}, 0) = 0.$$

If we find such an f, then we can go on and find in the next step another symplectic transformation which then eliminates the $\mathcal{O}(\varepsilon^2)$ terms, etc., until finally all perturbations are eliminated. Before we do so we look at the problem to eliminate the terms of order $\mathcal{O}(\varepsilon)$ in more detail. Given $H_1(\widetilde{\phi}, \widetilde{I}, 0)$ and $H_0(\widetilde{I})$ we seek $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, such that

$$\partial_{\widetilde{I}} H_0(\widetilde{I}) \cdot \partial_{\widetilde{\phi}} f(\widetilde{\phi}, \widetilde{I}) = -H_1(\widetilde{\phi}, \widetilde{I}, 0). \tag{4.17}$$

Since f is periodic in $\widetilde{\phi}$ we expand, for fixed $\widetilde{I}$, the problem into Fourier series, cf. §5.2.2. Thus,

$$f(\widetilde{\phi}) = \sum_{k \in \mathbb{Z}^d} a_k \mathrm{e}^{\mathrm{i}k \cdot \widetilde{\phi}}, \quad H_1(\widetilde{\phi}) = \sum_{k \in \mathbb{Z}^d} b_k \mathrm{e}^{\mathrm{i}k \cdot \widetilde{\phi}}, \quad k = (k_1, \dots, k_d).$$

Next we set

$$\omega(\widetilde{I}) = \partial_{\widetilde{I}} H_0(\widetilde{I}) = (\omega_1, \dots, \omega_d)(\widetilde{I}).$$

For $k \neq 0$ we obtain $\mathrm{i}(\omega \cdot k) a_k = -b_k$. Thus, if $\omega \cdot k \neq 0$, then

$$a_k = \frac{\mathrm{i} b_k}{\omega \cdot k},$$

is determined and the term $b_k \mathrm{e}^{\mathrm{i}k \cdot \widetilde{\phi}}$ can be removed from H_1. If there are no resonances at all, i.e., if $\omega \cdot k \neq 0$ for all $k \in \mathbb{Z} \setminus \{0\}$, then formally all $\mathcal{O}(\varepsilon)$ terms can be removed except of $b_0(\widetilde{I})$. However, this term can be included into $H_0(\widetilde{I})$ as a correction. Obviously the non-resonance condition for the elimination of the higher order terms is not changed since the left-hand side of (4.17) is not changed. If there are no resonances until the n^{th} step, then the perturbation up to terms of order $\mathcal{O}(\varepsilon^n)$ can be removed. We have the following approximation theorem.

Theorem 4.4.1. *If the normal form transformations allow to remove all terms up to order $\mathcal{O}(\varepsilon^n)$, i.e., if $\partial_{\widetilde{\phi}}\widetilde{H} = \mathcal{O}(\varepsilon^{n+1})$, then there exist C_1 and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have in the original coordinates*

$$\sup_{t\in(-1/\varepsilon^n, 1/\varepsilon^n)} \|I(t) - I(0)\| \leq C_1\varepsilon.$$

Proof. We have $\|I - \widetilde{I}(\phi, I)\| \leq C_2\varepsilon$ for a $C_2 > 0$, and, since $\|\dot{\widetilde{I}}\| = \|\partial_{\widetilde{\phi}}\widetilde{H}\| \leq C_3\varepsilon^{n+1}$ for a $C_3 > 0$, we have

$$\|\widetilde{I}(t) - \widetilde{I}(0)\| \leq C_3|t|\varepsilon^{n+1} \leq C_3\varepsilon$$

for all $|t| \leq 1/\varepsilon^n$. This yields

$$\begin{aligned}\|I(t) - I(0)\| \leq& \|I(t) - \widetilde{I}(t)\| + \|\widetilde{I}(t) - \widetilde{I}(0)\| + \|I(0) - \widetilde{I}(0)\| \\ \leq& (2C_2 + C_3)\varepsilon =: C_1\varepsilon.\end{aligned}$$

So far we did not consider the convergence of the above Fourier series in the normal form transforms. This turns out to be complicated due to so called small divisor problems. This means that for given $\omega \in \mathbb{R}^d$ and (arbitrary small) $\delta > 0$ there always is a $k \in \mathbb{Z}^d$ such that

$$|k \cdot \omega| < \delta.$$

Hence, the divisors in the series for f become arbitrarily small and the convergence of the Fourier series is a serious problem. The problem is solved by restricting the set of possible frequencies.

Definition 4.4.2. *A vector $\omega \in \mathbb{R}^d$ is called of type (L, γ) if for all $k \in \mathbb{Z}^d \setminus \{(0, \ldots, 0)\}$ we have*

$$|k \cdot \omega| \geq L|k|^{-\gamma}$$

We remark that for given $\gamma > d$ and almost all $\omega \in \mathbb{R}^d$ there exists a $L > 0$ such that ω is of type (L, γ), cf. [**Arn88**, page 114]. To study the analytic properties of the generating function F we use the following functions spaces.

Definition 4.4.3. *For $n \in \mathbb{N}$ define the spaces*

$$\ell_{1,n} = \{a : \mathbb{Z}^d \to \mathbb{C} : \|a\|_{\ell_{1,n}} = |a_0| + \sum_{k\in\mathbb{Z}^d} |a_k||k|^n < \infty\}.$$

Remark 4.4.4. We have that $a \in \ell_{1,n}$ implies $\mathcal{F}^{-1}a \in C_b^n$, where

$$C_b^n = \{f : \mathbb{T}^d \to \mathbb{R} : f\ n \text{ times continuously differentiable}\},$$

which is equipped with the norm $\|f\|_{C_b^n} = \sum_{|j|=0}^{n} \|\partial_\phi^j f\|_{C_b^0}$, where $\|f\|_{C_b^0} = \sup_{\phi\in\mathbb{T}^d} |f(\phi)|$, and where $\mathcal{F}^{-1} : a \mapsto f$ is defined by $f(\phi) = \sum_{k\in\mathbb{Z}^d} a_k \mathrm{e}^{\mathrm{i}k\cdot\phi}$, cf. §5.1. ⌋

Lemma 4.4.5. *Let $\omega \in \mathbb{R}^d$ be of type (L,γ) and $b \in \ell_{1,n}$ with $b_0 = 0$. Then a, defined by $a_k = \frac{ib_k}{k\cdot\omega}$ for $k \neq 0$, $a_0 = 0$, is in $\ell_{1,n-\gamma}$.*

Proof. We have

$$\|a\|_{\ell_{1,n-\gamma}} = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |a_k|k|^{n-\gamma}| = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \left|\frac{b_k}{k\cdot\omega}|k|^{n-\gamma}\right| \leq L^{-1}\|b\|_{\ell_{1,n}}. \quad \square$$

The correspondence of the exponent n in the weight of $\ell_{1,n}$ in Fourier space to regularity w.r.t. ϕ in physical space implies that in each iterative step we lose regularity of the Hamiltonian H. In order to eliminate the perturbation completely, infinitely many steps are necessary. Hence, there will be a loss of infinitely many derivatives. This problem can be solved by working in a space of analytic functions or by using some artificial smoothing in the so called hard implicit function theorem, cf. [**SR89**], when working with Hamiltonians of finite differentiability. The convergence is based on the quadratic convergence of the Newton scheme. This approach results in the famous **KAM**-theorem named after Kolmogorov, Arnold and Moser. We choose the analytic version and set

$$\mathcal{A}_{\sigma,\rho}(I^*) = \{(I,\phi) \in \mathbb{R}^n \times \mathbb{C}^n : |I - I^*| < \rho,\ |\mathrm{Im}(\phi_j)| < \sigma, j = 1,\ldots,d\}.$$

We define the norm of a function f which is analytic w.r.t. ϕ on $\mathcal{A}_{\sigma,\rho}(I^*)$ by

$$\|f\|_{\sigma,\rho} = \sup_{(I,\phi)\in\mathcal{A}_{\sigma,\rho}(I^*)} |f(I,\phi)|.$$

In nonlinear Hamiltonian systems in general the frequencies vary in a non-trivial manner with I, i.e., $\partial_I\omega = \partial_I^2 H_0$ does not vanish. Since there is a dense set of resonant frequencies in $\mathbb{R}^d$ it cannot be expected that the phase space is completely filled with tori after the perturbation. Therefore, in any neighborhood of a torus $\mathbb{T}^d$ with a non-resonant ω there is a torus $\mathbb{T}^d$ with a resonant ω. This means that next to any torus for which the transformations can be carried through to arbitrary order there is a torus in which low order perturbations influence the dynamics in the torus and may destroy the torus. Nevertheless, almost all tori persist in the following sense.

Theorem 4.4.6. (KAM) *Let $\omega(I^*) = \omega^*$ be of type (L,γ) and let the Hessian $\partial_I^2 H_0$ be invertible in I^*. Then there exists an $\varepsilon_0 > 0$ such that for $\|f\|_{\sigma,\rho} < \varepsilon_0$ the Hamiltonian system has quasi-periodic solutions with frequencies ω^*, i.e., the torus to $I = I^*$ persists under the perturbation. Let $V \subset \mathbb{R}^d$ be an open set with finite Lebesgue measure, where the Hessian $\partial_I^2 H_0$*

is invertible. For all $\delta > 0$ there is an $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there is a set $P_\varepsilon \subset V \times T^d$ with the following properties. The Lebesgue measure μ of $(V \times T^d) \setminus P_\varepsilon$ is less than δ, and for all $(I_0, \phi_0) \in P_\varepsilon$ the orbit through (I_0, ϕ_0) is quasi-periodic.

Proof. See [**Arn78, Way96**] or [**KP03**, §2] for a review. □

Hence, for small perturbations most of the phase space is still filled with tori. In between the tori chaotic behavior may occur. This is explained subsequently in §4.5.

In $\mathbb{R}^4$ the KAM-theorem yields a stability theorem since the invariant tori form two-dimensional hypersurfaces in the three-dimensional energy surfaces. In higher space dimensions the d-dimensional tori cannot separate domains in the $2d-1$-dimensional energy surface, but we expect that solutions need a long time to wander around the tori. This is called Arnold-diffusion and is mathematically formulated in Nekhoroshov's Theorem below.

An important motivation of these investigations again comes from celestial mechanics, in particular the question of the stability of our solar system. If we ignore mutual gravitational forces between the planets then we obtain a completely integrable system. The forces between the planets compared to that of the sun have a ratio of $\varepsilon \approx 1/1000$. Hence, the interplanetary forces can be considered as small perturbations. Quite obviously it is impossible to say whether our solar system is resonant or non-resonant, i.e., whether the ratios between different rotation times are rational or irrational. In spite of the fact that the rotation times of Jupiter and Saturn have a ratio of about $2/5$, our solar system seems to be remarkably stable. As said above, heuristically, even in the resonant case we expect the solutions to need a long time to wander around the remaining tori. This can be made precise for so called steep Hamiltonians, cf. [**AKN06**, §6.3.4].

Definition 4.4.7. *An analytic function is called steep if it has no real extrema and if all complex extrema are isolated.*

Theorem 4.4.8. (Nekhoroshov) *Let $H_0 = H_0(I)$ be a steep function. Then in the perturbed Hamiltonian system for a sufficiently small perturbation εH_1 we have*

$$\|I(t) - I(0)\|_{\mathbb{R}^d} < \varepsilon^b \tag{4.18}$$

for $0 \leq t \leq \frac{1}{\varepsilon}\exp(\frac{1}{c\varepsilon^a})$, where $a, b, c > 0$ only depend on H_0.

Remark 4.4.9. KAM-theory is used as an explanation for the so called Kirkwood gaps. Figure 4.4 shows the number of asteroids in the main asteroid belt as a function of their orbits major semi-axis in astronomical units (AU), where 1 AU is the length of the major semi-axis of the earth's orbit.

At certain values there are gaps in the distribution, and these correspond to low resonances between the periods of the asteroids and Jupiter. ⌋

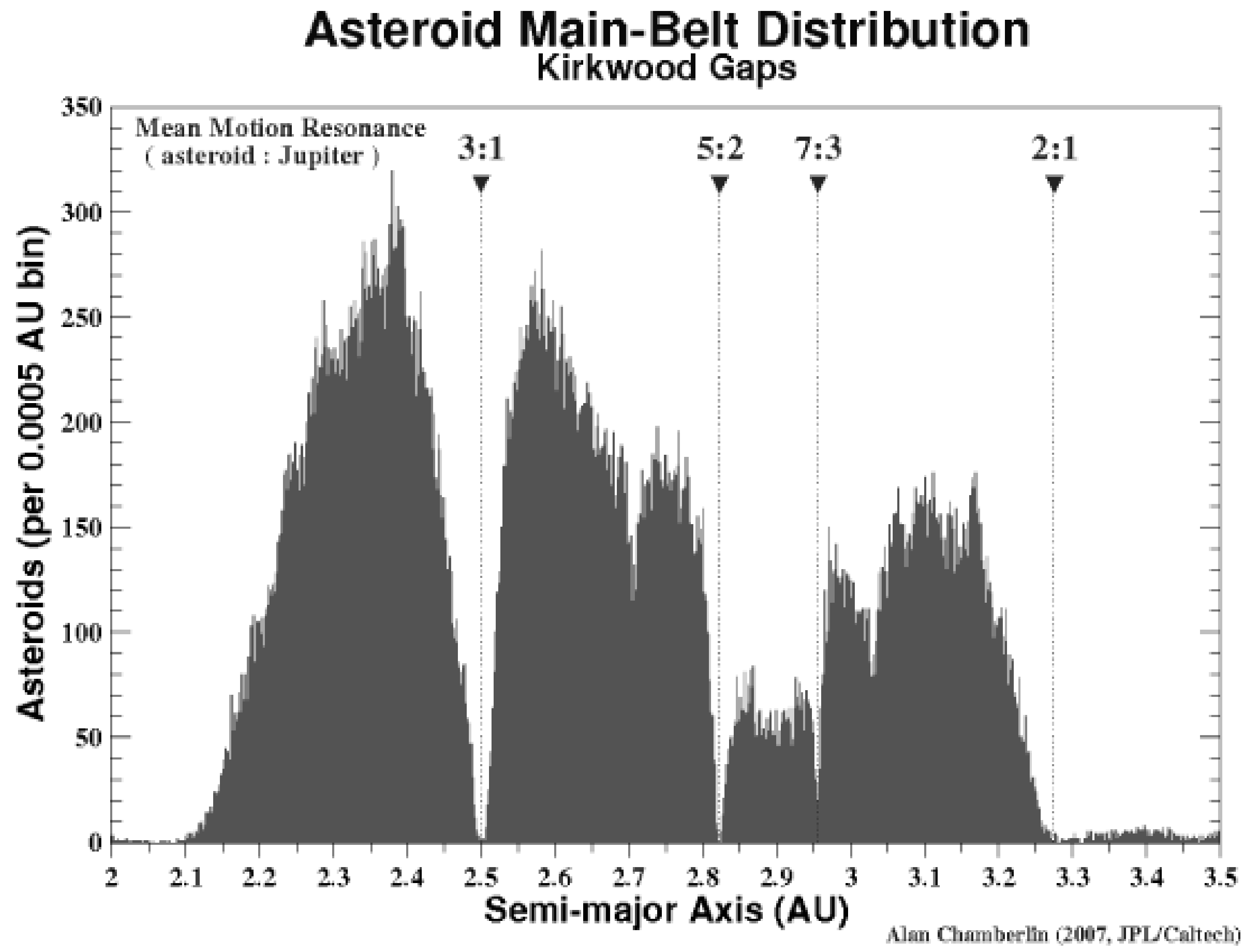


Figure 4.4. Kirkwood-gaps (ssd.jpl.nasa.gov/images/ast_histo.ps) Courtesy NASA/JPL-Caltech.

4.5. Homoclinic chaos

It is the purpose of this section to explain that in the part of the phase space which is not filled with invariant tori for small $\varepsilon > 0$ chaotic behavior can be expected. In between these tori there are periodic solutions and their stable and unstable manifolds. If there exists a heteroclinic connection with a transversal intersection of stable and unstable manifolds then a Smale horseshoe map and hence shift dynamics and chaotic behavior can be found. If for a map $f : \mathbb{R}^d \to \mathbb{R}^d$ the stable and unstable manifolds W_s and W_u of a hyperbolic fixed point p intersect transversally in a point q, then due to the invariance of the manifolds there must be infinitely many intersections. See Figure 4.5. Hence complicated dynamics can be expected. Recall that a fixed point p for an iteration $f : \mathbb{R}^d \to \mathbb{R}^d$ is called hyperbolic if the linearization $\partial_x f(p)$ possesses no eigenvalues on the unit circle.

Theorem 4.5.1. (The Smale-Birkhoff homoclinic orbit theorem) *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that p is a hyperbolic fixed point, and let $q \neq p$ be another point in which there is a transversal intersection of the stable manifold $W_s(p)$ and the unstable manifold $W_u(p)$. Then there*

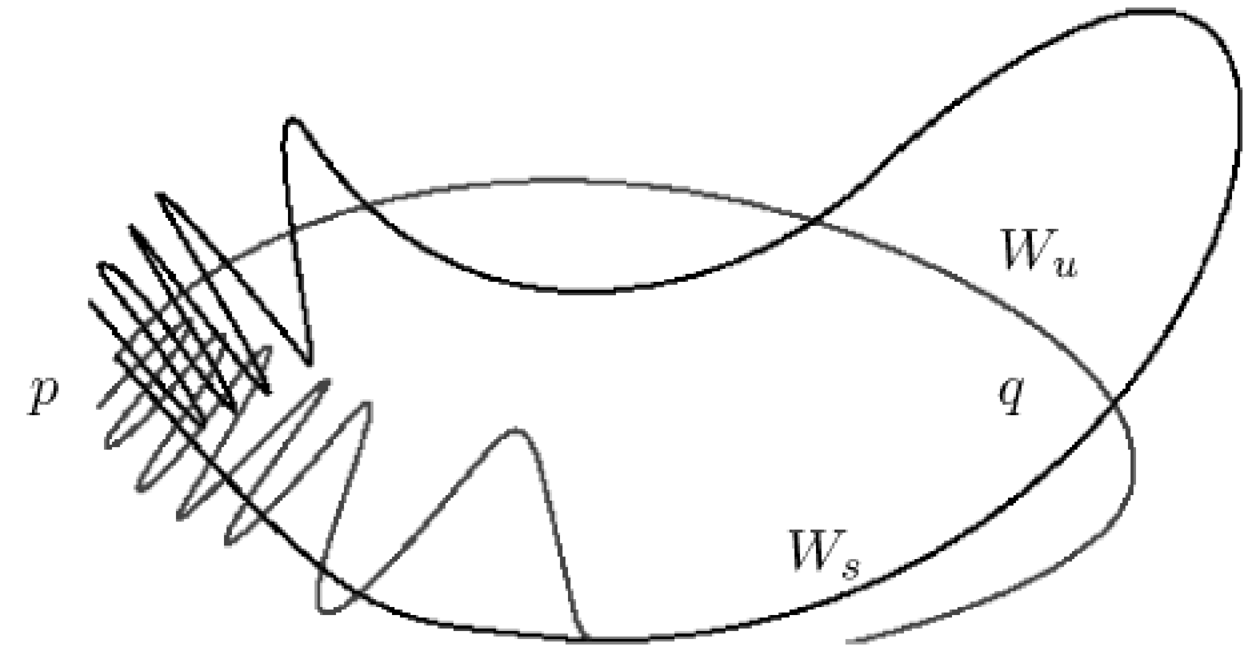


Figure 4.5. A transversal homoclinic point q implies infinitely many intersection points of the stable and unstable manifolds due to the invariance of the manifolds.

exists a (hyperbolic) set Λ on which an iteration of f is homeomorphic to shift dynamics.

Idea of the proof in $\mathbb{R}^2$: We are done if we find for an iteration of f a Smale's horseshoe, cf. Figure 2.14. W.l.o.g. let the saddle p be in the origin. By the Hartman-Grobman theorem, cf. Theorem 2.3.8, the saddle $(x, y) = (0, 0)$ has a neighborhood in which after some change of coordinates the dynamics is given by

$$x_{n+1} = \lambda x_n \quad \text{and} \quad y_{n+1} = \mu y_n$$

with $|\mu| > 1 > |\lambda|$. W.l.o.g. we can assume $\mu, \lambda > 0$. If this is not the case we consider the second iteration f^2. Then we consider

$$S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \delta,\ |y| \leq \delta\}$$

for $\delta > 0$ sufficiently small. The k^{th} iteration of f applied to S for k sufficiently large looks as sketched in Figure 4.6. Hence, we found a horseshoe in case of a homoclinic transversal point. □

This idea can be applied to 2π-time-periodic systems by considering the time 2π-map Π_ε. We derived from a $2d$-dimensional Hamiltonian system the $2(d-1)$-dimensional 2π-time-periodic Hamiltonian system (4.5)-(4.6). The d-dimensional tori break up and periodic solutions occur which are fixed points for the associated time 2π-map Π_ε. For a variety of systems numerical experiments indicate a transversal intersection of the associated stable and unstable manifolds and the occurrence of chaotic behavior between the persisting invariant tori, cf. [**Wig03**].

Remark 4.5.2. For time-periodic perturbations of an autonomous system with a homoclinic orbit the occurrence of a transversal intersection of the

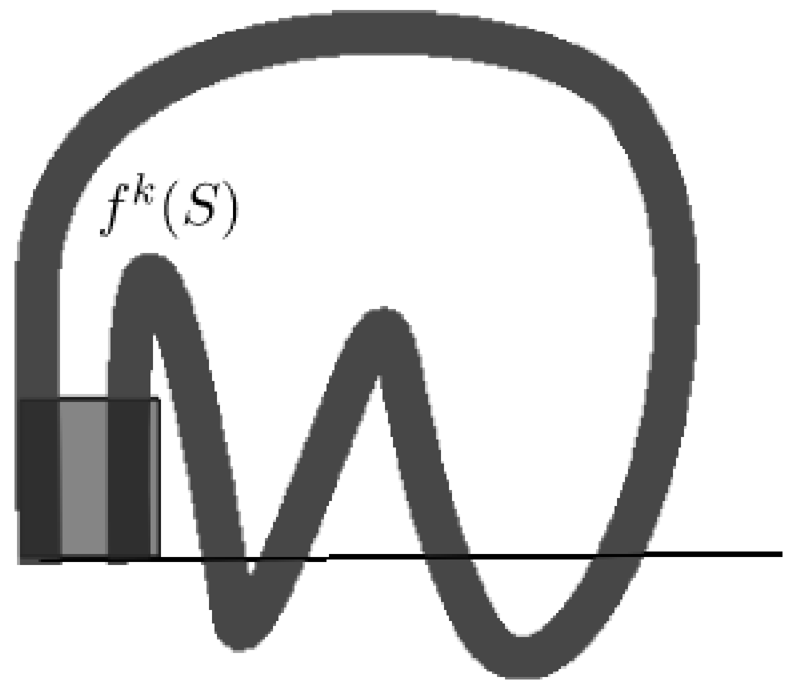


Figure 4.6. Smale's horseshoe in case of a homoclinic transversal point. The light gray rectangle is the set S and the dark gray set is $f^k(S)$.

stable and unstable manifolds can be established by finding single zeroes of the associated so called Melnikov function, cf. [**GH83**, §4.5]. ⌋

Exercises

4.1. Prove that $\dot{y} = (\text{trace}M)y$ for $y = \det Y$, where $Y(t) \in \mathbb{R}^{2\times 2}$ satisfies $\dot{Y} = MY$ for $M = M(t) \in \mathbb{R}^{2\times 2}$.

4.2. The "$6-12$ Lennart-Jones potential" models the forces between two neutral particles (atoms or molecules), namely an attractive van der Waals force at long ranges and a repulsive force at short ranges due to overlapping electron orbitals. In a simple (dimensionless) form it is given by $F(u) = au^{-12} - bu^{-6}$ where u is the distance between the particles and $a, b > 0$ are suitable constants. Choose $a = 0.001$ and $b = 1$ and discuss the phase portrait of the system $\ddot{u} = -F'(u)$.

4.3. Consider the pair $\ddot{x} + \omega^2 x = 0$, $\ddot{y} + \mu^2 y = 0$ of (uncoupled) harmonic oscillators. Write this as a Hamiltonian system. Find two integrals in polar coordinates. Discuss the cases (i) ω/μ rational and (ii) ω/μ irrational.

4.4. Given $r, \mu > 0$, write down explicitly a circular solution of the 1-body problem $\ddot{q} = -\mu q/\|q\|^3$, $q \in \mathbb{R}^2$, i.e., find initial conditions $q_0, \dot{q}_0$ such that the solution satisfies $\|q(t)\| = r$ for all $t \in \mathbb{R}$.

4.5. In dimensionless form, the first "Post-Newtonian" approximation for the orbit of a planet around the sun is

$$\partial_\theta^2 u + u = \alpha + \varepsilon u^2,$$

where $u = 1/r$ and (r, θ) are the polar coordinates of the planet and $\alpha, \varepsilon > 0$ are parameters. Discuss the phase portrait of this system.

4.6. Let $M \in \mathbb{R}^{n\times n}$ be nonsingular and symmetric and $F : \mathbb{R}^n \to \mathbb{R}$ be smooth. Write the Newtonian equation $M\ddot{x} + \nabla F(x) = 0$ as a Hamiltonian system.

4.7. Write the 4^{th} order ODE $u'''' + qu'' + f(u) = 0$ as a Hamiltonian system for (u, u', u'', u'''). *Hint.* Let $z=(u, u'')$ and derive a system $Tz''+\nabla V(z)=0$ with non-singular $T \in \mathbb{R}^{2\times 2}$.

4.8. Consider the 1-body problem $\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = J\nabla H(q,p), H(q,p) = U(q) + \frac{1}{2}|p|^2$. Show that the angular momentum $C = q \times \dot{q}$ is constant.

4.9. (The 2-body problem) Consider two mass points with positions $q_j \in \mathbb{R}^3$ and masses m_j that move under mutual gravitational attraction. The equations are

$$m_1\ddot{q}_1 = F_{21}, \quad m_2\ddot{q}_2 = F_{12},$$

with

$$F_{ij} = m_i m_j g(|q_i - q_j|)(q_i - q_j), \quad g(r) = G/r^3.$$

This problem can be completely reduced to the 1 body problem. For this consider the center of mass $q_s = (m_1q_1 + m_2q_2)/m_s$, with $m_s = m_1 + m_2$. Find the ODE for q_s and express the orbits $q_{1,2}$ via q_s and orbits of the 1 body problem for the distance $q = q_2 - q_1$.

4.10. Let $F, G, H : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth. Show that (a)

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

(b) F is an integral of $\dot{u} = J\nabla H$ iff $\{F, H\} = 0$. (c) $\frac{\mathrm{d}}{\mathrm{d}t}F(u) = \{F, H\}$.

4.11. Let $F = F(\widetilde{I}, \phi)$. Show that the map induced by $\widetilde{\phi} = \partial_{\widetilde{I}}F$ and $I = \partial_\phi F$ is symplectic.

Chapter 5

PDEs on an interval

The second part of this book is about nonlinear dynamics in countably many dimensions. It contains this chapter about PDEs on an interval and Chapter 6 about the Navier-Stokes equations.

We start with ordinary differential equations in $\mathbb{R}^\infty$, where $\mathbb{R}^\infty$ stands for the spaces $\mathbb{R}^\mathbb{N}$ or $\mathbb{R}^\mathbb{Z}$, i.e., for the spaces of real or complex (identifying $\mathbb{C}$ with $\mathbb{R}^2$) sequences $(a_j)_{j\in\mathbb{N}}$ or $(a_j)_{j\in\mathbb{Z}}$. In this book these countably many ODEs arise from PDEs, for which the spatial variable lives in a bounded domain. For function spaces on such domains very often a countable basis exists. By an expansion of the PDE w.r.t. this basis, for instance by an expansion into Fourier series in case of rectangular domains and suitable boundary conditions, the PDE can be transformed into an ODE in $\mathbb{R}^\infty$

Example. Consider the linear heat equation $\partial_t u = \partial_x^2 u$ for $x\in[0,\pi]$ with boundary condition $u(0,t)=u(\pi,t)=0$. Expanding

$$u(x,t)=\sum_{k\in\mathbb{N}} \widehat{u}_k(t)\sin(kx),$$

the PDE is formally equivalent to the countably many (uncoupled) ODEs $\frac{d}{dt}\widehat{u}_k=-k^2\widehat{u}_k$ for the sequence of Fourier coefficients $(\widehat{u}_k)_{k\in\mathbb{N}}$. ⌋

There are major differences between finitely and infinitely many dimensions due to the non-equivalence of norms in infinite-dimensional spaces and due to the loss of compactness of bounded closed sets. As a consequence, in infinite dimensions there can be stability w.r.t. one norm, but instability w.r.t. another norm. On the other hand there is a large class of equations, namely dissipative systems with smoothing properties, where the choice of the phase space does not matter. We however do not aim at a complete functional analytic treatment of such PDEs posed on bounded domains with dynamical systems concepts. For this we refer to the textbooks [**Hen81, Hal88, Rob01**].

Most dynamical system questions addressed in this part will be more involved when one considers PDEs on unbounded domains, as we do in Parts III and IV of this book. Hence, one of the main purposes of this section is to prepare for the additional difficulties as they appear in transferring the dynamical systems concept to PDEs posed on unbounded domains.

In §5.1 we consider the non-equivalence of norms, the loss and regain of compactness, and the local existence and uniqueness theory for countable many linear and nonlinear differential equations in $\mathbb{R}^\infty$. In §5.2 we discuss a number of basic function spaces, in particular those that are isomorphic via Fourier series to some sequence spaces. We explain local existence and uniqueness of solutions for some prototype linear and nonlinear PDEs, most of which will later be considered also over unbounded domains, and, moreover, explain how to prove global existence results. For these, the main tools are energy estimates and Gronwall type inequalities. We also give a characterization of the attractor of the so called Chafee-Infante problem, the scalar equation $\partial_t u = \partial_x^2 u + \alpha u - u^3$ on an interval $(0, \pi)$ with Dirichlet boundary conditions $u|_{x=0,\pi} = 0$, where $\alpha \in \mathbb{R}$ is a parameter.

5.1. From finitely to infinitely many dimensions

We consider systems of countably many linear and nonlinear differential equations. We discuss continuity of solutions w.r.t. time and some abstract local existence and uniqueness theory for ODEs in $\mathbb{R}^\infty$. Moreover, we explain how to differentiate and integrate in spaces of infinitely many dimensions and very briefly recall some basic facts from functional analysis, in particular compactness, which plays a crucial role for the dynamical systems point of view for PDEs

5.1.1. Non-equivalent norms. Concepts such as convergence in $\mathbb{R}^d$ or stability and instability for ODEs in $\mathbb{R}^d$ are independent of the chosen norm in $\mathbb{R}^d$. The reason for this is the equivalence of norms in finite-dimensional vector spaces, cf. Theorem 2.1.1. Setting $u = (u_1, \ldots, u_d)$, examples for norms in $\mathbb{R}^d$ have been

$$\|u\|_1 = \sum_{j=1}^{d} |u_j|, \qquad \|u\|_2 = \left(\sum_{j=1}^{d} |u_j|^2 \right)^{1/2},$$

and more generally $\|u\|_p := (\sum_{j=1}^d |u_j|^p)^{1/p}$, $1 \le p \le \infty$, and finally $\|u\|_\infty = \max_{j=1,\dots,d} |u_j|$. We have for instance

$$\|u\|_\infty \le \|u\|_p \le d^{1/p}\|u\|_\infty.$$

In infinite dimensions there are infinitely many non-equivalent norms. The norms which we use in this section are as follows.

Definition 5.1.1. *For $p \in [1,\infty)$ and $\theta \in \mathbb{R}$ let*

$$\|u\|_{\ell_{p,\theta}(\mathbb{R}^\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}} |u_n|^p \max(1,|n|)^{p\theta}\right)^{1/p}.$$

For $p = \infty$ and $\theta \in \mathbb{R}$ let

$$\|u\|_{\ell_{\infty,\theta}(\mathbb{R}^\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |u_n| \max(1,|n|)^{\theta}.$$

We set

$$\ell_{p,\theta}(\mathbb{R}^\mathbb{Z}) = \{u : \mathbb{Z} \to \mathbb{R} : \|u\|_{\ell_{p,\theta}(\mathbb{R}^\mathbb{Z})} < \infty\}.$$

Similarly, we define $\|\cdot\|_{\ell_{p,\theta}(\mathbb{R}^\mathbb{N})}$ and $\ell_{p,\theta}(\mathbb{R}^\mathbb{N})$. We use the abbreviations $\|\cdot\|_{\ell_{p,\theta}}$ and $\ell_{p,\theta}$ for $\|\cdot\|_{\ell_{p,\theta}(\mathbb{R}^\infty)}$ and $\ell_{p,\theta}(\mathbb{R}^\infty)$.

The norms for different p or different θ are not equivalent. As a consequence a sequence can converge in one norm towards 0 while it diverges to ∞ in another norm.

Example 5.1.2. For the sequence $(u^m)_{m\in\mathbb{N}}$, with $u^m \in \ell_{1,2}(\mathbb{R}^\mathbb{N})$ for fixed m defined through $u^m_n = \delta_{nm}/n$, we have $\|u^m\|_{\ell_{1,0}} = 1/m \to 0$ for $m \to \infty$, while $\|u^m\|_{\ell_{1,2}} = m \to \infty$ for $m \to \infty$. ⌋

Remark 5.1.3. The spaces $\ell_{p,\theta}$ are Banach spaces, i.e., complete normed vector spaces. We recall that a metric space M is called complete, if every Cauchy sequence in M possesses a limit in M. The spaces $\ell_{2,\theta}$ are Hilbert spaces, i.e., complete normed vector spaces where the norm is induced by a scalar product. The space $c_{00} = \{u : \mathbb{Z} \to \mathbb{R} : u_n \neq 0 \text{ for finitely many } n\}$ equipped with the ℓ_1 norm is not complete. See Exercise 5.1. ⌋

5.1.2. Linear differential equations in $\mathbb{R}^\infty$. For notational simplicity in the following we work with equations in $\mathbb{R}^\mathbb{N}$. The results for $\mathbb{R}^\mathbb{Z}$ are exactly the same. We consider linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}u = Au, \qquad \text{i.e.,} \qquad \frac{\mathrm{d}}{\mathrm{d}t}u_k = \sum_{j\in\mathbb{N}} a_{kj}u_j.$$

We briefly recall the basic notions of semigroup theory which is the abstract version of the subsequent analysis.

We are not interested in such equations in greatest generality and therefore restrict ourselves mainly to equations having to do with PDEs, i.e., we consider A in diagonal form or with Jordan blocks of finite size. In this situation the equation can be solved explicitly, but the analytic properties of the solutions still turn out to be rather subtle. In case of A in diagonal form

$$\frac{\mathrm{d}}{\mathrm{d}t} u_k = \lambda_k u_k$$

we find the solutions

$$u_k(t) = \mathrm{e}^{\lambda_k t} u_k(0).$$

Solutions $u(t)$ to linear differential equations in $\mathbb{R}^d$ with constant coefficients are arbitrarily smooth w.r.t. t. In $\mathbb{R}^{\mathbb{N}}$ this is no longer true. Even for the boundedness additional conditions are necessary.

Lemma 5.1.4. *Let* $\sup_{k\in\mathbb{N}} \mathrm{Re}\lambda_k = \alpha < \infty$. *Then for every* $\theta \in \mathbb{R}$, $T_0 > 0$, *and* $p \in [1, \infty]$ *the curve* $t \mapsto u(t)$ *is bounded in* $\ell_{p,\theta}$ *for* $t \in [0, T_0]$.

Proof. We have $\|u(t)\|_{\ell_{p,\theta}} \leq (\sup_{k\in\mathbb{N}} |\mathrm{e}^{\lambda_k t}|)\|u(0)\|_{\ell_{p,\theta}} \leq \mathrm{e}^{\alpha t}\|u(0)\|_{\ell_{p,\theta}}$. □

The next question is the continuity of the curve $t \mapsto u(t)$ in the spaces $\ell_{p,\theta}$. Which conditions do we have to impose on the eigenvalues λ_k to have continuity? We put this question into a bigger framework, namely the theory of semigroups. The solution operator $T(t) = \mathrm{diag}(\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t}, \ldots)$ defined through $T(t)u(0) = u(t)$ is an example of a semigroup of bounded linear operators, here from $\ell_{p,\theta}$ to $\ell_{p,\theta}$.

Definition 5.1.5. *Let* $(X, \|\cdot\|)$ *be a Banach space. A one-parameter family* $T(t)$, $0 \leq t < \infty$, *of bounded linear operators from* X *to* X *is called semigroup of bounded linear operators on* X, *if*

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$.

The linear operator $A : D(A) \to X$, *defined by*

$$D(A) = \{u \in X : \lim_{t\downarrow 0} \frac{T(t)u - u}{t} \text{ exists } \},$$

$$Au = \lim_{t\downarrow 0} \frac{T(t)u - u}{t}, \quad \text{for } u \in D(A),$$

is called the infinitesimal generator of $T(t)$.

According to the semigroup property, for the continuity of the maps $t \mapsto T(t)$ or $t \mapsto T(t)u$ the continuity at $t = 0$ is sufficient, cf. Remark 5.1.11. There are different concepts of continuity for semigroups. The first one is as follows.

Definition 5.1.6. *The semigroup is called operator-continuous if*

$$\lim_{t\downarrow 0} \|T(t) - I\| = 0$$

where $\|\cdot\|$ *denotes the operator norm.*

Example 5.1.7. Consider $\dot{u} = Au$ with $u(t) \in \mathbb{R}^d$ and A a $d \times d$-matrix with constant coefficients. Then $T(t) = \mathrm{e}^{tA}$ defines an operator-continuous semigroup in $\mathbb{R}^d$ since

$$\|\mathrm{e}^{tA} - I\| = \| \sum_{k=1}^{\infty} (At)^k/k!\| \leq |t|\|A\|\mathrm{e}^{|t|\|A\|} \to 0 \text{ for } t \to 0. \qquad \lrcorner$$

Theorem 5.1.8. *Let* $\sup_{k\in\mathbb{N}} |\lambda_k| = \alpha < \infty$*. Then for every* $\theta \in \mathbb{R}$ *and* $p \in [1, \infty]$ *the associated semigroup is operator-continuous in* $\ell_{p,\theta}$ *for all* $t \in \mathbb{R}$.

Proof. We have $\|u(t) - u(0)\|_{\ell_{p,\theta}} \leq (\sup_{k\in\mathbb{N}} |\mathrm{e}^{\lambda_k t} - 1|)\|u(0)\|_{\ell_{p,\theta}} \leq |\mathrm{e}^{\alpha t} - 1|\|u(0)\|_{\ell_{p,\theta}} \to 0$ for $t \to 0$. This implies $\|T(t) - I\|_{\ell_{p,\theta}\to\ell_{p,\theta}} \leq |\mathrm{e}^{\alpha t} - 1| \to 0$ for $t \to 0$ and so continuity holds. $\square$

For completeness we remark that a semigroup $T(t)$ of bounded linear operators on X is operator-continuous if and only if the generator $A : X \to X$ is bounded, cf. [**Paz83**, §1, Theorem 1.2]. Hence, as seen in the above example, the solutions of finite-dimensional ODEs always define an operator-continuous semigroup.

Since linearized operators in PDEs are usually unbounded, the generated semigroups are in general only strongly continuous.

Definition 5.1.9. *A semigroup* $T(t)$*,* $0 \leq t < \infty$*, of bounded linear operators in* X *is called strongly continuous semigroup, or* C_0*-semigroup, if*

$$\lim_{t\downarrow 0} \|T(t)u - u\| = 0 \qquad \textit{for each} \quad u \in X.$$

Theorem 5.1.10. *For every* $\theta \in \mathbb{R}$*,* $p \in [1, \infty)$*, and* $u(0) \in \ell_{p,\theta}$*, the curve* $t \mapsto u(t)$ *is continuous in* $\ell_{p,\theta}$ *for* $t \geq 0$ *if and only if* $\sup_{j\in\mathbb{N}} \mathrm{Re}\lambda_j = \alpha < \infty$.

Proof. Let $\varepsilon > 0$. Using the triangle inequality in $\ell_{p,\theta}$ we have that

$$\begin{aligned} &\|u(t) - u(0)\|_{\ell_{p,\theta}} \\ &= (\sum_{n=1}^{N} |(\mathrm{e}^{\lambda_n t} - 1)u_n(0)|^p |n|^{p\theta})^{1/p} + (\sum_{n=N+1}^{\infty} |(\mathrm{e}^{\lambda_n t} - 1)u_n(0)|^p |n|^{p\theta})^{1/p} \\ &= s_1 + s_2 \end{aligned}$$

for a $N \in \mathbb{N}$ suitably chosen in the following. In order to prove that $s_1 + s_2 < \varepsilon$ for $t > 0$ sufficiently small we first estimate s_2 by choosing N so big that

$$s_2 \leq (\mathrm{e}^{\alpha t} + 1)(\sum_{n=N+1}^{\infty} |u_n(0)|^p |n|^{p\theta})^{1/p} < \varepsilon/2.$$

For this N we then find a $t_0 > 0$ such that for all $t \in (0, t_0)$ we can estimate

$$s_1 \leq (\max_{n=1,\ldots,N} |\mathrm{e}^{\lambda_n t} - 1|)\|u(0)\|_{\ell^{p,\theta}} < \varepsilon/2.$$

Therefore, we are done. $\square$

Remark 5.1.11. Since $T(t_0 + h)u - T(t_0)u = (T(h) - I)T(t_0)u \to 0$ and $T(t_0 - h)u - T(t_0)u = -T(t_0 - h)(T(h) - I)u \to 0$ for $h \downarrow 0$ the right-continuity in $t_0 = 0$ implies the continuity in every $t_0 > 0$ if the semigroup is uniformly bounded on every compact interval, cf. Lemma 5.1.4. In fact the assumption of the uniform boundedness on every compact interval is satisfied for C_0-semigroups due to a deep result from functional analysis, namely the uniform boundedness principle, cf. [**Paz83**, §1.2, Theorem 2.2]. $\rfloor$

For $\dot{u} = Au$, with A in diagonal form, for $p \in [1, \infty)$ every bounded trajectory is also continuous in t. However, there is no uniformity w.r.t. the initial conditions $u(0)$. In $\ell_{\infty,\theta}$ the assumption about the boundedness of the eigenvalues λ_j is also necessary for continuity as the following example shows.

Example 5.1.12. Let $\lambda_k = -k^2$. Then there exists an $u(0) \in \ell_{\infty,0}$, for instance $u(0) = (1, 1, 1, \ldots)$, such that $\|u(t) - u(0)\|_{\ell_{\infty,0}} = 1$ for every $t > 0$ such that continuity cannot hold. $\rfloor$

Solutions of ODEs in $\mathbb{R}^d$ are smooth if the data is smooth. In $\mathbb{R}^\infty$ for $u(t) = (\mathrm{e}^{\lambda_n t} u_n(0))_{n \in \mathbb{N}}$ additional conditions about the eigenvalues λ_n are necessary.

- The m^{th} derivative $u^{(m)}(t)$ is given by $u_n^{(m)}(t) = \lambda_n^m \mathrm{e}^{\lambda_n t} u_n(0)$. For $u(0) \in \ell_{p,\theta}$ we can guarantee the m-times differentiability of $t \mapsto u(t)$ in $\ell_{p,\theta}$ if the eigenvalues are in a set

$$\{\lambda \in \mathbb{C} : t\mathrm{Re}\lambda \leq a - m \ln |\mathrm{Im}\lambda|\}$$

for some constant $a \in \mathbb{R}$, cf. [**Paz83**, §2.4, Theorem 4.8]. As an example we consider $\lambda_n = -\ln n + in$. In order to have the m-times differentiability we need that $u^{(m)}(t) \in \ell_{p,\theta}$ which follows if

$$\sup_{n \in \mathbb{N}} |(-\ln n + in)^m \mathrm{e}^{-(\ln n)t}| = \sup_{n \in \mathbb{N}} |(-\ln n + in)^m n^{-t}| < \infty.$$

This means that the curve is one time differentiable for $t \in (1, 2]$, two times differentiable for $t \in (2, 3]$, etc.

- For $u(0) \in \ell_{p,\theta}$ we have the analyticity of $t \mapsto u(t)$ if the eigenvalues are in a sector
$$S_{a,b} = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq a - b \ln |\mathrm{Im}\lambda|\}$$
for some constants $a \in \mathbb{R}$ and $b \geq 0$, cf. [**Paz83**, §2.5, Theorem 5.2]. As an example we consider $\lambda_n = -n + (-1)^n in$. The function $t \mapsto \mathrm{e}^{\lambda_n t} u_n(0)$ can be extended analytically into a sector of the complex plane. For $t = t_r + it_i$ we find $u(t_r + it_i) \in \ell_{p,\theta}$ if
$$\sup_{n\in\mathbb{N}} |\mathrm{e}^{(-n+(-1)^n in)(t_r+it_i)}| \leq \sup_{n\in\mathbb{N}} |\mathrm{e}^{n(-t_r+|t_i|)}| < \infty$$
which holds if $|t_i| < t_r$. Such generators are called sectorial and play a major role in the analysis of dissipative systems. The associated semigroup $(\mathrm{e}^{\lambda_n t})_{n\in\mathbb{N}}$ is called analytic. We come back to this in §6.3.

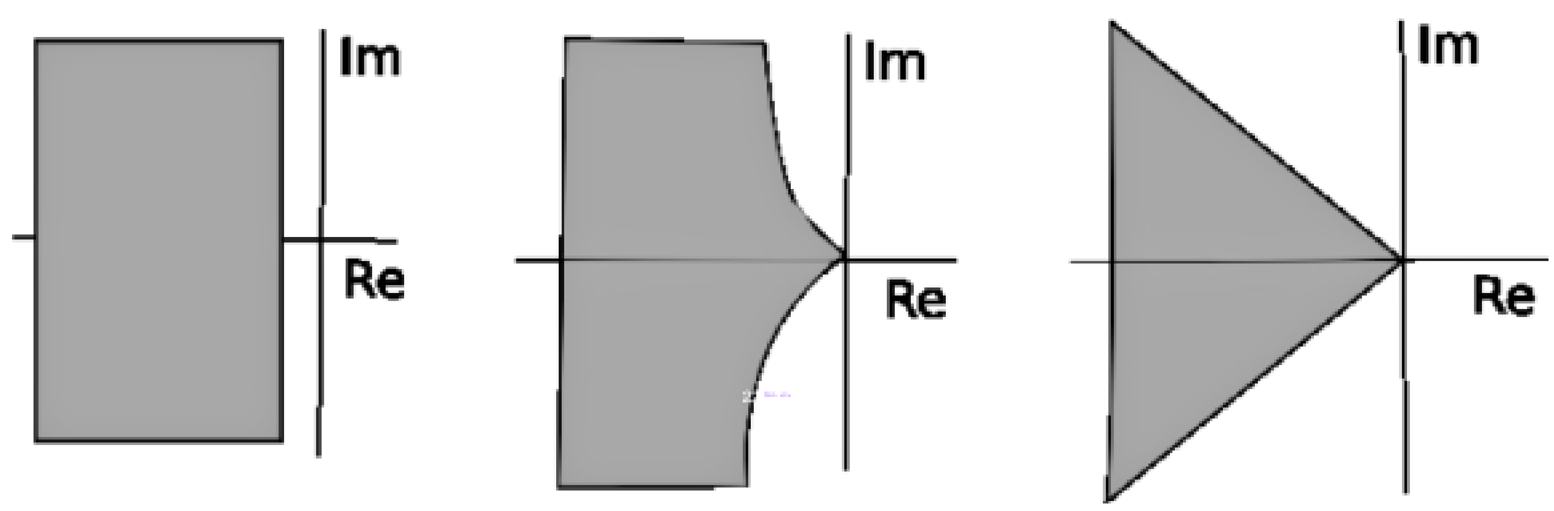


Figure 5.1. The picture shows the regions where the spectrum of the generators must be contained in to have a continuous (left panel), a differentiable (middle panel), or an analytic (right panel) semigroup.

We refer to the textbook [**Paz83**] for a thorough introduction to semigroup theory. Generators of C_0-semigroups are characterized by the theorem of Hille-Yosida, cf. [**Paz83**, §1.3, Theorem 3.1] or the Lumer-Phillips theorem, cf. [**Paz83**, §1.4, Theorem 4.3].

5.1.3. Differentiation and integration in Banach spaces. Before we proceed with the consideration of nonlinear infinite-dimensional ODEs, we need some additional functional analytic tools. For the stability of fixed points in ODEs the linearization, i.e., the derivative $A = Df \in \mathbb{R}^{d\times d}$ of a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ plays a central role. Hence, the concept of derivatives has to be generalized from $\mathbb{R}^d$ to general Banach spaces X. The same is true for integration. In the iteration scheme used in the proof of the local existence and uniqueness theorem for ODEs, a continuous function on an interval with values in $\mathbb{R}^d$ is integrated. If this iteration scheme is transferred to PDEs, then a continuous function on an interval with values in some infinite-dimensional Banach space has to be integrated. Such integrations occur in other iteration schemes used for PDEs, too. Hence, we have to define the integral of a continuous function on an interval with values in some Banach space. It turns out that the usual definition with Riemann sums is sufficient for our purposes.

Let us start with the derivatives. The Gateaux derivative is a generalization of the concept of the directional derivative. In Banach spaces the derivative is also called Fréchet derivative, cf. [**AA11**].

Definition 5.1.13. *Suppose that X and Y are Banach spaces, that $U \subset X$ is open, and consider $F : X \to Y$. The Gâteaux derivative $DF(u)[v]$ of F at $u \in U$ in the direction $v \in X$ is defined as*

$$DF(u)[v] = \lim_{\tau \to 0} \frac{F(u + \tau v) - F(u)}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} F(u + \tau v)\bigg|_{\tau=0}.$$

If the limit exists for all $v \in X$, then F is called Gâteaux differentiable at u. $F : X \to Y$ is called differentiable in $u \in U$ if there exists a bounded linear operator $A = A(u) : X \to Y$ such that

$$\lim_{h \to 0} \frac{\|F(u + h) - F(u) - A(u)h\|_Y}{\|h\|_X} = 0.$$

The function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1, & \text{if } y = x^2,\ x \neq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is a finite-dimensional example of a function for which every directional derivative exists, but which is not differentiable. In infinite-dimensional spaces less 'exotic' examples are possible.

Example 5.1.14. Consider $X = Y = L^2(0, 1)$ and $F(u)(x) = \sin(u(x))$. We show that F is Gâteaux differentiable, but not differentiable at $u = 0$. We have

$$\lim_{\tau \to 0} \frac{F(u + \tau v) - F(u)}{\tau} = \lim_{\tau \to 0} \frac{\sin(\tau v(x))}{\tau} = \cos(0)v(x) = v(x)$$

due to the differentiability of $\sin : \mathbb{R} \to \mathbb{R}$. For the Fréchet differentiability we can vary v not only along lines. Due to the above computed Gâteaux derivative the only possible candidate for the derivative $A(0)$ is the identity. We set

$$v_n(x) = \begin{cases} n\pi, & \text{if } x \in (0, 1/n^4) \\ 0, & \text{elsewhere} \end{cases}$$

and find

$$\frac{\|F(v_n) - F(0) - A(0)v_n\|_Y}{\|v_n\|_X} = \frac{\|v_n\|_Y}{\|v_n\|_X} = 1 \not\to 0$$

although $\|v_n\|_{L^2} = \pi/n \to 0$ for $n \to \infty$. We remark that with the choice $X = Y = C_b^0([0, 1])$ equipped with the sup-norm the map F would be analytic. ⌋

However, in spaces with additional algebra properties, virtually all results from complex power series carry over.

Example 5.1.15. Let $(X, \|\cdot\|)$ be a Banach space. Then $Y = \mathcal{L}(X, X) = \{F : X \to X : \text{linear, continuous}\}$ is a Banach space equipped with the operator norm

$$\|A\| = \sup\{\|Au\|_X \ : \ \|u\|_X = 1\}.$$

For $A, B \in Y$ we have that AB and A^n are in Y with $\|AB\| \le \|A\|\|B\|$, and $\|A^n\| \le \|A\|^n$. Therefore, the series $F(A) = \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ is convergent in Y. We have the continuity and the Fréchet differentiability of $F : Y \to Y$ with $DF(A) = F(A)$. The analyticity of F follows like for real-valued power series, cf. Exercise 5.3. ⌋

Next we come to the integration of continuous functions $f : [a, b] \to X$ with values in a Banach space X. Let $P = \{x_0, \dots, x_n\}$ with $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a partition of the interval. Its fineness is defined by

$$\|P\| = \max\{|x_{j+1} - x_j| : j = 0, \dots, n-1\}.$$

Let $\xi = (\xi_1, \dots, \xi_n)$ with $\xi_j \in [x_{j-1}, x_j]$. Then define the Riemann sum

$$S(\xi, P) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}).$$

Definition 5.1.16. *A function $f : [a, b] \to X$ is called Riemann integrable if the limit*

$$\lim_{n\to\infty} S(\xi(n), P(n))$$

exists for every sequence $(\xi(n), P(n))$ with $\lim_{n\to\infty} \|P(n)\| = 0$. If the limit exists, then we define the Riemann integral by

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|P\|\to 0} S(\xi, P).$$

Theorem 5.1.17. *Continuous functions $f : [a, b] \to X$ are Riemann integrable.*

Proof. We have to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all (ξ^1, P_1), (ξ^2, P_2) with $\|P_1\| \le \delta$ and $\|P_2\| \le \delta$ we have $\|S(\xi^1, P_1) - S(\xi^2, P_2)\|_X \le \varepsilon$.

We set $P_3 = P_1 \cup P_2$ and choose an arbitrary ξ^3. Then by the triangle inequality we have

$$\begin{aligned}\|S(\xi^1, P_1) - S(\xi^2, P_2)\|_X \le& \|S(\xi^1, P_1) - S(\xi^3, P_3)\|_X + \|S(\xi^3, P_3 - S(\xi^2, P_2)\|_X \\ \le& \varepsilon/2 + \varepsilon/2 \le \varepsilon.\end{aligned}$$

where we used

$$\|S(\xi^1, P_1) - S(\xi^3, P_3)\|_X = \| \sum_{j=1}^{N_1} f(\xi_j^1)(\widetilde{x}_j - \widetilde{x}_{j-1}) - \sum_{j=1}^{N_3} f(\xi_j^3)(x_j - x_{j-1}) \|_X$$

$$(5.1) \qquad \leq \sum_{j=1}^{N_1} \sum_{k=\alpha(j-1)}^{\alpha(j)-1} \left\| f(\xi_j^1) - f(\xi_{k+1}^3) \right\|_X |x_{k+1} - x_k| \,,$$

where $[\widetilde{x}_{j-1}, \widetilde{x}_j] = \bigcup_{k=\alpha(j-1)}^{\alpha(j)-1} [x_k, x_{k+1}]$. See Figure 5.2.

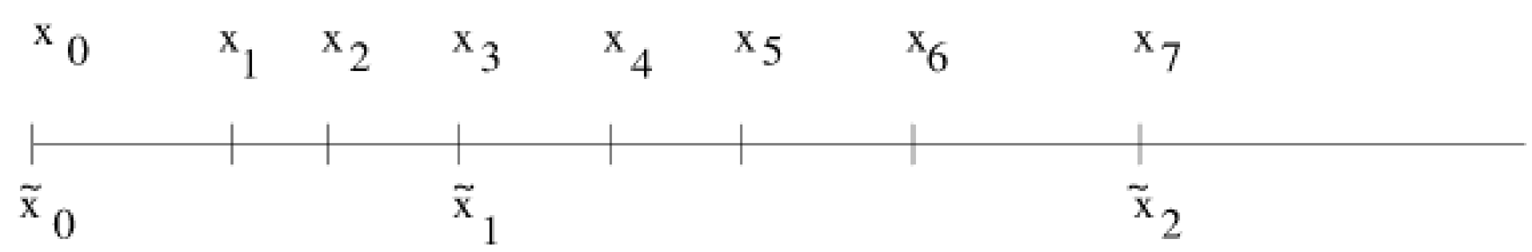


Figure 5.2. The partition P_1 is drawn below the line and P_3 above the line. In this example we have $\alpha(0) = 0$, $\alpha(1) = 3$, $\alpha(2) = 7, \ldots$.

By uniform continuity of f, which follows from the continuity of f on the compact interval $[a, b]$, we have that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|y - \widetilde{y}| < \delta$ implies $\|f(y) - f(\widetilde{y})\|_X < \frac{\varepsilon}{2(b-a)}$. Hence, if $\|P_3\| \leq \|P_1\| < \delta$ is chosen sufficiently small, (5.1) can be estimated by

$$\leq \sum_{k=1}^{N_3} \frac{\varepsilon}{2(b-a)} |x_k - x_{k-1}| \leq \frac{\varepsilon}{2}. \qquad \square$$

Remark 5.1.18. Not only the Riemann integral can be generalized to functions $u : \mathbb{R} \to X$, with X some Banach space, but also the Lebesgue integral, cf. [**Alt16**, §A1]. ⌋

5.1.4. Nonlinear differential equations in $\mathbb{R}^\infty$. Since in general the ODEs in $\mathbb{R}^\infty$ obtained from PDEs have unbounded λ_ks the right-hand side is no longer Lipschitz-continuous from $\ell_{p,\theta}$ to $\ell_{p,\theta}$. Thus, the Picard-Lindelöf theorem no longer applies and has to be replaced. The simplest idea to obtain a contraction as in the proof of the Picard-Lindelöf theorem is the use of the variation of constant formula, cf. §2.1.3.

For simplicity we first consider

$$(5.2) \qquad \frac{\mathrm{d}}{\mathrm{d}t} u = \Lambda u + B(u, u),$$

where $u(t) \in \ell_{p,\theta}$, where Λ is a diagonal matrix with entries λ_k satisfying

$$(5.3) \qquad \sup_k \mathrm{Re} \lambda_k = \beta < \infty,$$

and where B is a bilinear symmetric map from $\ell_{p,\theta}$ into $\ell_{p,\theta}$ satisfying

$$(5.4) \qquad \|B(u, v)\|_{\ell_{p,\theta}} \leq C_B \|u\|_{\ell_{p,\theta}} \|v\|_{\ell_{p,\theta}}.$$

In order to prove the local existence and uniqueness of solutions of (5.2) on an interval $[0, T_0]$ we use the variation of constant formula to rewrite (5.2) into

$$u(t) = \mathrm{e}^{t\Lambda}u(0) + \int_0^t \mathrm{e}^{(t-\tau)\Lambda} B(u(\tau), u(\tau))\,\mathrm{d}\tau =: F(u)(t). \tag{5.5}$$

Definition 5.1.19. *a) A function $u \in C^0([0, T_0], \ell_{p,\theta})$ which satisfies* (5.5) *is called a mild solution of* (5.2).
b) A function $u \in C^1([0, T_0], \ell_{p,\theta})$, with $\Lambda u \in C([0, T_0], \ell_{p,\theta})$, is called a strong solution of (5.2), *if* (5.2) *holds in $\ell_{p,\theta}$ for every $t \in (0, T_0)$.*

Clearly, every strong solution is a mild solution. Conversely, a mild solution which satisfies $u \in C^1([0, T_0], \ell_{p,\theta})$ and $\Lambda u \in C([0, T_0], \ell_{p,\theta})$ is a strong solution. In the following until further notice solutions will always mean mild solutions.

Theorem 5.1.20. *Assume* (5.3) *and* (5.4). *For all $C_1 > 0$ there exists a $T_0 > 0$ such that for all $w \in \ell_{p,\theta}$ with $\|w\|_{\ell_{p,\theta}} \leq C_1$ we have a unique solution $u \in C([0, T_0], \ell_{p,\theta})$ of* (5.2) *with initial condition $u(0) = w$.*

Proof. We fix a $C_2 > 0$ and show that for $T_0 \in (0, 1)$ sufficiently small the right-hand side of (5.5) is a contraction in the set

$$M = C([0, T_0], \{u(t) \in \ell_{p,\theta} : \|u(t) - \mathrm{e}^{t\Lambda}u(0)\|_{\ell_{p,\theta}} \leq C_2\}),$$

and apply the contraction mapping theorem. M is a complete metric space, but since the metric is induced by a norm we will use the norm notation in the following. We use the abbreviation $C_3 = \sup_{u \in M} \|u\|_M \leq C_1 e^{\beta} + C_2$.

In a first step we prove that F maps M into itself. We have

$$\begin{aligned}
\|F(u) - (\mathrm{e}^{t\Lambda}u(0))_{t\geq 0}\|_M &= \sup_{t\in[0,T_0]} \|F(u)(t) - \mathrm{e}^{\Lambda t}u(0)\|_{\ell_{p,\theta}} \\
&\leq \sup_{t\in[0,T_0]} \| \int_0^t \mathrm{e}^{(t-\tau)\Lambda} B(u(\tau), u(\tau))\,\mathrm{d}\tau \|_{\ell_{p,\theta}} \\
&\leq \sup_{t\in[0,T_0]} \int_0^t \mathrm{e}^{\beta(t-\tau)} \|B(u(\tau), u(\tau))\|_{\ell_{p,\theta}}\,\mathrm{d}\tau \\
&\leq T_0 \mathrm{e}^{\beta T_0} C_B C_3^2 \leq C_1
\end{aligned}$$

for $T_0 > 0$ sufficiently small.

Secondly, we find that F is a contraction since

$$\begin{aligned}
\|F(u)-F(v)\|_M &= \sup_{t\in[0,T_0]} \|F(u)(t)-F(v)(t)\|_{\ell_{p,\theta}} \\
&\le \sup_{t\in[0,T_0]} \Big\| \int_0^t \mathrm{e}^{(t-\tau)\Lambda}(B(u(\tau),u(\tau))-B(v(\tau),v(\tau)))\,\mathrm{d}\tau\Big\|_{\ell_{p,\theta}} \\
&\le \sup_{t\in[0,T_0]} \int_0^t \mathrm{e}^{\beta(t-\tau)}\|B(u(\tau),u(\tau))-B(v(\tau),v(\tau))\|_{\ell_{p,\theta}}\,\mathrm{d}\tau \\
&\le T_0\mathrm{e}^{\beta T_0} \sup_{\tau\in[0,T_0]} \|B(u(\tau)+v(\tau),u(\tau)-v(\tau))\|_{\ell_{p,\theta}} \\
&\le 2T_0\mathrm{e}^{\beta T_0}C_BC_3\|u-v\|_M \le \|u-v\|_M/2
\end{aligned}$$

for $T_0>0$ sufficiently small. □

This procedure of constructing solutions to ODEs in $\mathbb{R}^\infty$ can be extended to a wider class of problems. We consider again

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \Lambda u + B(u,u), \tag{5.6}$$

where $u(t)\in\ell_{p,\theta}$, but now with the following assumptions:

- Λ a diagonal matrix satisfying

$$\|\mathrm{e}^{t\Lambda}u\|_{\ell_{p,\theta}} \le C_{\theta-r}\mathrm{e}^{\beta t}t^{-\alpha}\|u\|_{\ell_{p,r}} \tag{5.7}$$

  for an $\alpha\in[0,1)$, a constant $C_{\theta-r}$, and $\theta-r\ge 0$.

- B a bilinear symmetric map from $\ell_{p,\theta}$ into $\ell_{p,r}$ satisfying

$$\|B(u,v)\|_{\ell_{p,r}} \le C_B\|u\|_{\ell_{p,\theta}}\|v\|_{\ell_{p,\theta}}. \tag{5.8}$$

The property described by equation (5.7) is called smoothing since the evolution operator maps for $t>0$ the space $\ell_{p,r}$ into $\ell_{p,\theta}$ and since functions whose Fourier coefficients are in $\ell_{p,\theta}$ are smoother than functions whose Fourier coefficients are only $\ell_{p,r}$. Many of the subsequent examples will satisfy estimates like (5.7).

Example 5.1.21. Consider $\lambda_n=-n^2$. We have the decay estimate

$$\|(\mathrm{e}^{\lambda_n t}u_n)_{n\in\mathbb{N}}\|_{\ell_{p,\theta}} \le \sup_{n\in\mathbb{N}}|\mathrm{e}^{-n^2t}n^\theta|\|u\|_{\ell_{p,0}} \le Ct^{-\theta/2}\|u\|_{\ell_{p,0}},$$

which corresponds to smoothing of functions in physical space, see Example 5.2.19. ⌋

Remark 5.1.22. Smoothing is not directly related to regularity w.r.t. time t, as the following examples show. In case $\lambda_n=0$ for all $n\in\mathbb{N}$ all eigenvalues are identical and contained in a sector. However, the associated semigroup is the identity which is not smoothing from $\ell_{p,r}$ into $\ell_{p,\theta}$ for $r<\theta$ although we have an analytic semigroup (w.r.t. time). In case $\lambda_n=-n^2+\mathrm{i}(-1)^n n^3$

obviously the eigenvalues are not contained in a sector and the semigroup is not analytic (w.r.t. time $t = t_r + \mathrm{i}t_i$) since

$$\sup_{n\in\mathbb{N}} |\mathrm{e}^{(-n^2+\mathrm{i}(-1)^n n^3)(t_r+it_i)}| = \sup_{n\in\mathbb{N}} |\mathrm{e}^{-n^2 t_r+(-1)^{n+1}n^3 t_i}| = \infty$$

for $t_i \neq 0$. However, we have the decay estimate

$$\|(\mathrm{e}^{\lambda_n t}u_n)_{n\in\mathbb{N}}\|_{\ell_{p,\theta}} \leq \sup_{n\in\mathbb{N}} |\mathrm{e}^{(-n^2\pm\mathrm{i}n^3)t}n^\theta|\|u\|_{\ell_{p,0}} \leq Ct^{-\theta/2}\|u\|_{\ell_{p,0}}. \qquad \rfloor$$

In order to prove the local existence and uniqueness of solutions of (5.6) on an interval $[0, T_0]$, we again use the variation of constant formula

$$u(t) = \mathrm{e}^{t\Lambda}u(0) + \int_0^t \mathrm{e}^{(t-\tau)\Lambda}B(u(\tau), u(\tau))\,\mathrm{d}\tau =: F(u)(t). \tag{5.9}$$

We show that for $T_0 \in (0, 1)$ sufficiently small the right-hand side of (5.9) is a contraction in the set

$$M = C([0, T_0], \{u(t) \in \ell_{p,\theta} : \|u(t) - \mathrm{e}^{t\Lambda}u(0)\|_{\ell_{p,\theta}} \leq C_2\}),$$

where $\|u(0)\|_{\ell_{p,\theta}} \leq C_1$ and $C_2 > 0$ is a fixed constant. In a first step we prove that F maps M into itself. With C_3 as above we have

$$\begin{aligned}
\|F(u)-(\mathrm{e}^{t\Lambda}u(0))_{t\geq 0}\|_M &= \sup_{t\in[0,T_0]} \|F(u)(t) - \mathrm{e}^{t\Lambda}u(0)\|_{\ell_{p,\theta}} \\
&\leq \sup_{t\in[0,T_0]} \|\int_0^t \mathrm{e}^{(t-\tau)\Lambda}B(u(\tau), u(\tau))\,\mathrm{d}\tau\|_{\ell_{p,\theta}} \\
&\leq \sup_{t\in[0,T_0]} C_{\theta-r}\int_0^t (t-\tau)^{-\alpha}\mathrm{e}^{\beta(t-\tau)}\|B(u(\tau), u(\tau))\|_{\ell_{p,r}}\,\mathrm{d}\tau \\
&\leq C_{\theta-r}(1-\alpha)^{-1}T_0^{1-\alpha}\mathrm{e}^{\beta T_0}C_B C_3^2 \leq C_1
\end{aligned}$$

for $T_0 > 0$ sufficiently small.

Secondly, we find

$$\begin{aligned}
\|F(u) - F(v)\|_M &= \sup_{t\in[0,T_0]} \|F(u)(t) - F(v)(t)\|_{\ell_{p,\theta}} \\
&\leq \sup_{t\in[0,T_0]} \|\int_0^t \mathrm{e}^{(t-\tau)\Lambda}(B(u(\tau), u(\tau)) - B(v(\tau), v(\tau)))\,\mathrm{d}\tau\|_{\ell_{p,\theta}} \\
&\leq \sup_{t\in[0,T_0]} \int_0^t C_{\theta-r}(t-\tau)^{-\alpha}\mathrm{e}^{\beta(t-\tau)}\|B(u(\tau), u(\tau)) - B(v(\tau), v(\tau))\|_{\ell_{p,r}}\,\mathrm{d}\tau \\
&\leq C_{\theta-r}(1-\alpha)^{-1}T_0^{1-\alpha}\mathrm{e}^{\beta T_0} \sup_{t\in[0,T_0]} \|B(u(\tau) + v(\tau), u(\tau) - v(\tau))\|_{\ell_{p,\theta}} \\
&\leq 2C_{\theta-r}(1-\alpha)^{-1}T_0^{1-\alpha}\mathrm{e}^{\beta T_0}C_B C_3\|u - v\|_M \leq \|u - v\|_M/2
\end{aligned}$$

for $T_0 > 0$ sufficiently small. Hence, the contraction F possesses a unique fixed point in M and so we have proved

Theorem 5.1.23. *Assume* (5.7) *and* (5.8). *For all* $C_1 > 0$ *there exists a* $T_0 > 0$ *such that for all* $w \in \ell_{p,\theta}$ *with* $\|w\|_{\ell_{p,\theta}} \leq C_1$ *we have a unique solution* $u \in C([0, T_0], \ell_{p,\theta})$ *of* (5.6) *with initial condition* $u(0) = w$.

Remark 5.1.24. Both theorems obviously also hold if B is replaced by a general locally Lipschitz-continuous map from $\ell_{p,\theta}$ into $\ell_{p,\theta}$ or $\ell_{p,r}$ respectively, i.e., for instance in the latter case that for all C_1 there exists an L such that $\max(\|u\|_{\ell_{p,\theta}}, \|v\|_{\ell_{p,\theta}}) \leq C_1$ implies

$$\|N(u) - N(v)\|_{\ell_{p,r}} \leq L\|u - v\|_{\ell_{p,\theta}}.$$

Every polynomial nonlinearity is locally Lipschitz-continuous in this sense.⌋

Moreover, Theorem 5.1.20 and Theorem 5.1.23 are prototypes for other local existence and uniqueness theorems for semi-linear evolutionary PDEs below.

5.1.5. A first look on Fourier series. PDEs with periodic boundary conditions for the spatial coordinates can be transferred to ODEs in $\mathbb{R}^\infty$ with the help of Fourier series. Here we give the definition and some elementary properties. For later purposes we consider here the d-dimensional situation.

Definition 5.1.25. *A series of the form*

$$u(x) = \sum_{k \in \mathbb{Z}^d} \widehat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{5.10}$$

is called Fourier series, its partial sums $u(x) = \sum_{|k| \leq N} \widehat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}$, *are called Fourier polynomials of order* N, *and* $\widehat{u}_k$ *is called the* k^{th} *Fourier coefficient.*

See §5.2.2 for more details, in particular a number of convergence results for (5.10). Since our main interest is in nonlinear PDEs we also have to handle products of functions in physical space with Fourier series. The point-wise multiplication in physical space correspond in Fourier space to convolution. That is,

$$u(x)v(x) = \left(\sum_{k \in \mathbb{Z}^d} \widehat{u}_k \mathrm{e}^{ik\cdot x} \right) \left(\sum_{m \in \mathbb{Z}^d} \widehat{v}_m \mathrm{e}^{im\cdot x} \right) = \sum_{k \in \mathbb{Z}^d} \left(\sum_{m \in \mathbb{Z}^d} \widehat{u}_{k-m} \widehat{v}_m \right) \mathrm{e}^{ik\cdot x}.$$

This motivates the definition of the convolution

$$(\widehat{u} * \widehat{v})_k = \sum_{m \in \mathbb{Z}^d} \widehat{u}_{k-m} \widehat{v}_m.$$

For the control of the nonlinear terms in Fourier space we need

Lemma 5.1.26. (Young's inequality for convolutions) *For* $p \in [1, \infty]$ *we have*

$$\|\widehat{u} * \widehat{v}\|_{\ell_p} \leq \|\widehat{u}\|_{\ell_p} \|\widehat{v}\|_{\ell_1}.$$

Proof. For $p \in [1,\infty]$ we find

$$\begin{aligned}\|\widehat{u}*\widehat{v}\|_{\ell_p} =&(\sum_{k\in\mathbb{Z}}|\sum_{m\in\mathbb{Z}}\widehat{u}_{k-m}\widehat{v}_m|^p)^{1/p} \le (\sum_{l\in\mathbb{Z}}(\sum_{m\in\mathbb{Z}}|\widehat{u}_l\widehat{v}_m|)^p)^{1/p}\\ =&(\sum_{l\in\mathbb{Z}}|\widehat{u}_l|^p(\sum_{m\in\mathbb{Z}}|\widehat{v}_m|)^p)^{1/p} \le \|\widehat{u}\|_{\ell_p}\|\widehat{v}\|_{\ell_1}.\end{aligned}$$

The case $p=\infty$ is obvious. □

Young's inequality for convolutions allows us to prove that the $\ell_{2,\theta}$-spaces are closed under convolution if θ is sufficiently big. In order to do so we prove the following version of Sobolev's embedding theorem

Lemma 5.1.27. *For $m-d/2>n$ there exists a $C>0$, such that*

$$\|\widehat{u}\|_{\ell_{1,n}(\mathbb{R}^d)} \le C\|\widehat{u}\|_{\ell_{2,m}(\mathbb{R}^d)}.$$

Proof. With $\widehat{\rho}_k=\max(1,|k|)$ the estimate follows from

$$\begin{aligned}\|\widehat{u}\|_{\ell_{1,n}} &= \sum_{k\in\mathbb{Z}^d}|\widehat{u}_k|\widehat{\rho}_k^n = \sum_{k\in\mathbb{Z}^d}|\widehat{u}_k|\widehat{\rho}_k^m\widehat{\rho}_k^{(n-m)}\\ &\le \left(\sum_{k\in\mathbb{Z}^d}|\widehat{u}_k|^2\widehat{\rho}_k^{2m}\right)^{1/2}\left(\sum_{k\in\mathbb{Z}^d}\widehat{\rho}_k^{2(n-m)}\right)^{1/2} \le C\|\widehat{u}\|_{\ell_{2,m}},\end{aligned}$$

since $\sum_{k\in\mathbb{Z}^d}\widehat{\rho}_k^{2(n-m)}<\infty$, due to $m-d/2>n$ by assumption. □

We use this embedding to establish

Lemma 5.1.28. *a) For all $m\ge 0$ there exists a $C>0$, such that for all $\widehat{u},\widehat{v}\in\ell_{1,m}$ we have*

$$\|\widehat{u}*\widehat{v}\|_{\ell_{1,m}} \le C\|\widehat{u}\|_{\ell_{1,m}}\|\widehat{v}\|_{\ell_{1,m}}.$$

b) For all $m>d/2$ there exists a $C>0$, such that for all $\widehat{u},\widehat{v}\in\ell_{2,m}$ we have

$$\|\widehat{u}*\widehat{v}\|_{\ell_{2,m}} \le C\|\widehat{u}\|_{\ell_{2,m}}\|\widehat{v}\|_{\ell_{2,m}}.$$

Proof. a) Since $\widehat{\rho}_k^m \le C(\widehat{\rho}_{k-l}^m+\widehat{\rho}_l^m)$ with $C=2^m$ for $\widehat{\rho}_k=\max(1,|k|)$ using Lemma 5.1.26 it follows that

$$\begin{aligned}\|\widehat{u}*\widehat{v}\|_{\ell_{1,m}} =& \sum_{k\in\mathbb{Z}^d}|\sum_{l\in\mathbb{Z}^d}\widehat{u}_{k-l}\widehat{v}_l\widehat{\rho}_k^m| \le \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|\widehat{u}_{k-l}\widehat{v}_l\widehat{\rho}_k^m|\\ \le& C\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}(|\widehat{u}_k\widehat{v}_l|\widehat{\rho}_k^m+|\widehat{u}_k\widehat{v}_l|\widehat{\rho}_l^m)\\ \le& C(\|\widehat{u}\|_{\ell_{1,0}}\|\widehat{v}\|_{\ell_{1,m}}+\|\widehat{u}\|_{\ell_{1,m}}\|\widehat{v}\|_{\ell_{1,0}}) \le 2C\|\widehat{u}\|_{\ell_{1,m}}\|\widehat{v}\|_{\ell_{1,m}}.\end{aligned}$$

b) With Lemma 5.1.26 we have

$$\begin{aligned}
\|\widehat{u} * \widehat{v}\|_{\ell_{2,m}} &= \Big(\sum_{k\in\mathbb{Z}^d} \Big|\sum_{l\in\mathbb{Z}^d} \widehat{u}_{k-l}\widehat{v}_l\widehat{\rho}_k^m\Big|^2\Big)^{1/2} \\
&\leq \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{l\in\mathbb{Z}^d} |\widehat{u}_{k-l}\widehat{v}_l|(\widehat{\rho}_{k-l}^m + \widehat{\rho}_l^m)\Big)^2\Big)^{1/2} \\
&\leq C(\||\widehat{u}|\widehat{\rho}^m * |\widehat{v}|\|_{\ell_{2,0}} + \||\widehat{u}| * |\widehat{v}|\widehat{\rho}^m\|_{\ell_{2,0}}) \\
&\leq C(\|\widehat{u}\|_{\ell_{2,m}}\|\widehat{v}\|_{\ell_{1,0}} + \|\widehat{u}\|_{\ell_{1,0}}\|\widehat{v}\|_{\ell_{2,m}}).
\end{aligned}$$

The final assertion follows from Sobolev's embedding theorem $\|\widehat{v}\|_{\ell_{1,0}} \leq C\|\widehat{v}\|_{\ell_{2,m}}$ for $m > d/2$. □

We now give a number of classical examples of nonlinear PDEs over intervals with periodic boundary conditions. In fact, over unbounded domains each of these equations will play an important role in this book. For the modeling and physical background of the equations we refer in particular to Part III.

Example 5.1.29. Let $u(x,t) = u(x+2\pi,t) \in \mathbb{R}$, $A(X,T) = A(X+2\pi,T) \in \mathbb{C}$, and $\widehat{u}_k(t)$, $\widehat{A}_k(T)$ be the associated Fourier coefficients.

a) The Kolmogorov, Petrovsky, Piskounov (KPP) equation $\partial_t u = \partial_x^2 u + u - u^2$ transforms into

$$\partial_t\widehat{u}_k = -k^2\widehat{u}_k + \widehat{u}_k - \sum_{m\in\mathbb{Z}} \widehat{u}_{k-m}\widehat{u}_m.$$

b) The Allen-Cahn equation $\partial_t u = \partial_x^2 u + u - u^3$ transforms into

$$\partial_t\widehat{u}_k = -k^2\widehat{u}_k + \widehat{u}_k - \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \widehat{u}_{k-m}\widehat{u}_{m-l}\widehat{u}_l.$$

c) The Burgers equation $\partial_t u = \partial_x^2 u + \partial_x(u^2)$ transforms into

$$\partial_t\widehat{u}_k = -k^2\widehat{u}_k + \mathrm{i}k\sum_{m\in\mathbb{Z}} \widehat{u}_{k-m}\widehat{u}_m.$$

d) The Korteweg-deVries (KdV) equation $\partial_t u = \partial_x^3 u + \partial_x(u^2)$ transforms into

$$\partial_t\widehat{u}_k = -\mathrm{i}k^3\widehat{u}_k + \mathrm{i}k\sum_{m\in\mathbb{Z}} \widehat{u}_{k-m}\widehat{u}_m.$$

e) Using $(\mathcal{F}\overline{A})_j = \overline{\widehat{A}}_{-j}$ the Nonlinear Schrödinger (NLS) equation $\partial_T A = \mathrm{i}\partial_X^2 A + \mathrm{i}|A|^2 A$ transforms into

$$\partial_T\widehat{A}_k = -\mathrm{i}k^2\widehat{A}_k + \mathrm{i}\sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \widehat{A}_{k+l-m}\overline{\widehat{A}}_l\widehat{A}_m.$$

f) The complex Ginzburg-Landau (GL) equation $\partial_T A = (1+\mathrm{i}\alpha)\partial_X^2 A + RA - (1+\mathrm{i}\beta)|A|^2 A$, with $\alpha, \beta, R \in \mathbb{R}$, transforms into

$$\partial_T \widehat{A}_k = -(1+\mathrm{i}\alpha)k^2 \widehat{A}_k + \widehat{A}_k - (1+\mathrm{i}\beta) \sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \widehat{A}_{k+l-m} \overline{\widehat{A}}_l \widehat{A}_m.$$

Except for the KdV equation, for all equations from above the local existence and uniqueness theory can be handled with Theorem 5.1.20 and Theorem 5.1.23. The linear parts are given by the eigenvalues λ_k, $k \in \mathbb{Z}$, where

$$\begin{aligned} &\text{a), b) } \lambda_k = -k^2 + 1, \quad \text{c) } \lambda_k = -k^2, \\ &\text{d) } \lambda_k = -\mathrm{i}k^3, \quad \text{e)} \lambda_k = -\mathrm{i}k^2, \quad \text{f) } \lambda_k = -(1+\mathrm{i}\mu)k^2 + 1. \end{aligned}$$

In a), b), e) and f) we only need that $(\mathrm{e}^{\lambda_k t})_{k\in\mathbb{Z}} : \ell_{2,\theta} \to \ell_{2,\theta}$ is bounded for fixed t. Since the nonlinear terms in a), b), e) and f) are bi- and trilinear maps from $\ell_{2,\theta} \to \ell_{2,\theta}$ for $\theta > 1/2$ we have the local existence and uniqueness for these equations in $\ell_{2,\theta}$ for $\theta > 1/2$ according to Theorem 5.1.20. Since for c) and d) the nonlinear terms are only bilinear maps from $\ell_{p,\theta+1}$ into $\ell_{p,\theta}$ we need an estimate

$$\|(\mathrm{e}^{\lambda_k t})_{k\in\mathbb{Z}}\|_{\ell_{p,\theta}\to\ell_{p,s+1}} \leq C \max(1, t^{-\alpha}) \tag{5.11}$$

with $\alpha \in [0,1)$ for the semigroup in order to apply our local existence and uniqueness result from Theorem 5.1.23. According to Example 5.1.21 such an estimate is true for c) with $\alpha = 1/2$, but not for d). The KdV equation is a so called a quasilinear (hyperbolic) equation. There is local existence in $\ell_{2,\theta}$ for $\theta = 3$, for instance. However, the proof is more involved, cf. [**Paz83**, §8, Theorem 5.6] or §8.2 for further remarks. ⌋

5.1.6. Loss and regain of compactness. We close this section with a number of comments on compactness, which is a crucial concept to define attractors in dynamical systems. In metric spaces there are the following equivalent characterizations of compact sets, cf. [**Alt16**, §2.5].

Definition 5.1.30. *Let (M,d) be a complete metric space.*

a) A set $A \subset M$ is compact if every covering of A by open sets contains a finite subcovering.

b) A set $A \subset M$ is (sequentially) compact if every sequence in A has a convergent subsequence with limit in A.

c) A set $A \subset M$ is compact if A is closed and pre-compact, where a set $A \subset M$ is said to be pre-compact if for every $\varepsilon > 0$, there exists a finite subset $\{s_1, s_2, \ldots, s_n\}$ of A such that $A \subset \bigcup_{k=1}^n B(s_k, \varepsilon)$, where $B(s_k, \varepsilon)$ denotes the open ball around s_k with radius ε.

Compactness arguments in the sense of b) were used a number of times in Part I. Examples are the existence, respectively the non-emptyness, of ω-limit sets and attractors. For the term 'pre-compact' also the term 'totally bounded' is used in the literature. In $\mathbb{R}^d$ compact sets can be characterized by the theorem of Heine-Borel.

Theorem 5.1.31. *In $\mathbb{R}^d$ a set is compact if and only if it is closed and bounded.*

In infinite-dimensional spaces compactness is more restrictive due to the fact that the theorem of Heine-Borel is no longer true as the following example shows.

Example 5.1.32. Consider the closed unit ball in $(\mathbb{R}^{\mathbb{N}}, \|\cdot\|_\infty)$. The sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n = e_n$ satisfies $\|u_n - u_m\|_\infty = \delta_{nm}$ such that no convergent subsequence can exist. Variants of this example works in all norms from above such that the closed unit ball is not compact in any of the norms from Definition 5.1.1. ⌋

The equivalence of compactness to boundedness and closedness is a precise distinction between finite- and infinite-dimensional Banach spaces [**Alt16**, Satz 2.9]. There are famous theorems about the characterization of pre-compact subsets of function spaces. These are the Arzela-Ascoli theorem [**Alt16**, Satz 2.11], the theorem of Riesz [**Alt16**, Satz 2.15], and Sobolev's embedding theorem [**Alt16**, Satz 8.9].

Compactness in infinite-dimensional spaces will be regained by smoothing properties of the evolution operators. For instance the evolution operator of Example 5.1.21 maps bounded balls of $\ell_{2,0}$ into bounded balls of $\ell_{2,1}$ for every fixed $t > 0$. Since the subsequent version of Sobolev's embedding theorem 5.1.33 guarantees that bounded balls of $\ell_{2,1}$ are pre-compact sets of $\ell_{2,0}$, the evolution operator of Example 5.1.21 maps bounded balls of $\ell_{2,0}$ into pre-compact sets of $\ell_{2,0}$. This property will be used for showing that ω-limit sets and attractors for such systems are non-empty. The following theorem is also known under the name Rellich's embedding theorem.

Theorem 5.1.33. *The space $\ell_{p,\theta}$ can be compactly embedded into the space $\ell_{p,r}$ for all $p \geq 1$ and $\theta > r$.*

Proof. For notational simplicity we restrict to the index set $\mathbb{N}$. Compactly embedded means that every bounded set of $\ell_{p,\theta}$ is pre-compact in $\ell_{p,r}$. Due to the homogeneity of the spaces it is sufficient to prove that the unit ball of $\ell_{p,\theta}$ can be covered by finitely many balls of $\ell_{p,r}$ with radius ε. In order to do so we consider the first n_0 coordinates. The restriction of the unit ball of $\ell_{p,\theta}$ to these coordinates is a pre-compact set in $\mathbb{R}^{n_0}$. Hence, for every $\varepsilon > 0$ it can be covered by finitely many balls $B_\varepsilon(z_j)$ of $\mathbb{R}^{n_0}$ w.r.t. the $\ell_{p,r}$-norm and with $z_j \in \mathbb{R}^{n_0}$ for $j = 1, \dots, N$. We claim that the unit ball of $\ell_{p,\theta}$ is contained in the union of the balls $B_{\varepsilon/2}((z_j, 0))$ of $\ell_{p,r}(\mathbb{R}^{\mathbb{N}})$ if we choose $(n_0+1)^{r-\theta} \le \varepsilon/2$. This follows since for $u = ((u_k)_{k=1,\dots,n_0}, u_\infty)$ in the unit ball of $\ell_{p,\theta}(\mathbb{R}^{\mathbb{N}})$ we have a $j \in \{1, \dots, N\}$ such that

$$\|u-(z_j,0)\|_{\ell_{p,r}} \le \|((u_k)_{k=1,\dots,n_0},0)-(z_j,0)\|_{\ell_{p,r}}+\|(0,u_\infty)\|_{\ell_{p,r}} \le \varepsilon/2+\varepsilon/2=\varepsilon,$$

where $\|((u_k)_{k=1,\dots,n_0}, 0) - (z_j, 0)\|_{\ell_{p,r}} \le \varepsilon/2$ due to the construction of the points z_j and where $\|(0, u_\infty)\|_{\ell_{p,r}} \le \varepsilon/2$ due to

$$\begin{aligned}\|(0,u_\infty)\|_{\ell_{p,r}} \le& \Big(\sum_{n_0+1}^{\infty} |u_n|^p |n|^{pr}\Big)^{1/p} \le \sup_{n=n_0+1,\dots,\infty} |n|^{r-\theta} \Big(\sum_{n_0+1}^{\infty} |u_n|^p |n|^{p\theta}\Big)^{1/p} \\ \le& (n_0+1)^{r-\theta} \|(0,u_\infty)\|_{\ell_{p,\theta}} \le \varepsilon/2\end{aligned}$$

for n_0 sufficiently large since $\|(0, u_\infty)\|_{\ell_{p,\theta}} \le 1$. □

5.2. Basic function spaces and Fourier series

PDEs posed on spatially bounded domains are very often isomorphic to ODEs in $\mathbb{R}^\infty$. Thus, the abstract set-up from the last section can often be applied to solve PDEs posed on spatially bounded domains. However, a big part of PDE theory is concerned with problems coming from the boundary of the considered domains. These play almost no role in this book, i.e., they are circumvented by considering almost all systems subsequently with periodic boundary conditions or on the real line. This allows us to concentrate on phenomena coming from the equations. In other words, a complete functional analytic treatment of PDEs posed on bounded domains with dynamical systems concepts is beyond the scope of this book. For this we refer to the textbooks [**Hen81, Hal88, Tem97**]. However, in Part IV of this book some of the methods to handle problems posed on cylindrical domains $\mathbb{R} \times \Sigma$, with $\Sigma \subset \mathbb{R}^d$ a bounded domain, are explained. In this section we concentrate on PDEs where the spatial coordinate lives on a bounded interval with periodic boundary conditions. Such problems can easily be related to ODEs in $\mathbb{R}^\infty$ with the help of Fourier series. These explanations are embedded in some theoretical background about basic function spaces and Fourier series.

5.2.1. Basic function spaces. The solution $u = u(\cdot, t)$ of a PDE is for fixed t in some function space. Here, we introduce some basic function spaces following [**Alt16, Wlo87**]. In the following let $\Omega \subset \mathbb{R}^d$ be an open set, $x = (x_1, \dots, x_d) \in \Omega$, $n = (n_1, \dots, n_d)$ a multi-index, $|n| = n_1 + \dots + n_d$, and $\partial_x^n = \partial_{x_1}^{n_1} \dots \partial_{x_d}^{n_d}$.

Continuous and differentiable functions. The space of continuous functions in $\overline{\Omega}$ is

$$C^0(\overline{\Omega},\mathbb{R}) = \{u : \overline{\Omega} \to \mathbb{R} : u \text{ is continuous}\},$$

equipped with the norm

$$\|u\|_{C_b^0} = \sup_{x\in\overline{\Omega}} |u(x)|.$$

The space of m-times continuously differentiable functions in $\overline{\Omega}$ is

$$C^m(\overline{\Omega},\mathbb{R}) = \{u : \overline{\Omega} \to \mathbb{R} : \partial_x^j u \text{ is continuous for } |j| = 0,\dots,m\},$$

equipped with the norm

$$\|u\|_{C_b^m} = \sum_{0\le|j|\le m} \|\partial_x^j u\|_{C_b^0}.$$

From the definition it is clear that for $u \in C^m(\overline{\Omega},\mathbb{R})$ we have $\|u\|_{C_b^m} < \infty$, if Ω is bounded. More generally, we define

$$C_b^m(\overline{\Omega},\mathbb{R}) = \{u \in C^m(\overline{\Omega},\mathbb{R}) : \|u\|_{C_b^m} < \infty\}.$$

For $\Omega = \overline{\Omega} = \mathbb{R}$ the function $u(x) = x$ is in C^0, but not in C_b^0. For Ω bounded, C_b^m is dense in C_b^0. For the treatment of unbounded Ω we define

$$\begin{aligned} C_{b,unif}^m(\overline{\Omega},\mathbb{R}) = \{u : \overline{\Omega} \to \mathbb{R} :&\partial_x^j u \text{ is uniformly continuous for} \\ & |j| = 0,\dots,m, \|u\|_{C_b^m} < \infty\}. \end{aligned}$$

For $\Omega = \mathbb{R}$ the function $u(x) = \sin(x^2)$ is in C_b^0, but not in $C_{b,unif}^0$. For similar reasons $C_b^n(\mathbb{R},\mathbb{R})$ is not dense in $C_b^0(\mathbb{R},\mathbb{R})$, but $C_{b,unif}^n(\mathbb{R},\mathbb{R})$ in $C_{b,unif}^0(\mathbb{R},\mathbb{R})$. All these spaces are Banach spaces.

Hölder spaces. The spaces C_b^0 and C_b^m are not the optimal choice for solving linear PDEs. Even for arbitrarily smooth boundary $\partial\Omega$ the boundary value problem

$$\Delta u = f \qquad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{5.12}$$

for $f \in C_b^0$ in general does not possess a solution u with optimal regularity, cf. Example 5.2.4 on page 157. Optimal regularity holds for the subsequently defined Hölder-continuous functions and Sobolev functions, i.e., for instance for (5.12) from $f \in C^{0,\alpha}$ it follows $u \in C^{2,\alpha}$. For $\alpha \in (0,1]$ we define

$$C^{0,\alpha}(\overline{\Omega},\mathbb{R}) = \{u : \overline{\Omega} \to \mathbb{R} : u \;\text{ is } \alpha\text{ -Hölder-continuous, } \|u\|_{C^{0,\alpha}} < \infty\}$$

equipped with the norm

$$\|u\|_{C^{0,\alpha}} = \|u\|_{C_b^0} + \sup_{x,y\in\overline{\Omega},x\neq y,|x-y|\le 1} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

and, for $k \in \mathbb{N}$ and $\alpha \in (0,1]$,

$$\begin{aligned} C^{k,\alpha}(\overline{\Omega},\mathbb{R}) =& \{u : \overline{\Omega} \to \mathbb{R} : \partial_x^j u \in C^0 \text{ for } |j| = 0,\dots,k, \\ & \qquad \partial_x^k u \in C^{0,\alpha},\ \|u\|_{C^{k,\alpha}} < \infty\} \end{aligned}$$

equipped with the norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C_b^{k-1}} + \sum_{|j|=k} \|\partial_x^j u\|_{C^{0,\alpha}}.$$

All these function spaces are Banach spaces, cf. Exercise 5.9. $C^{0,1}(\overline{\Omega},\mathbb{R})$ is the space of Lipschitz-continuous functions.

Lebesgue and Sobolev spaces. Unfortunately, the above spaces are not equipped with a scalar product and so tools from linear algebra related to orthogonality are not available. A natural choice of a scalar product for functions would be

$$\langle u, v\rangle_{L^2} = \int_\Omega u(x)\overline{v(x)}\,\mathrm{d}x. \tag{5.13}$$

However, if the above spaces are equipped with the above scalar product they are not complete w.r.t. the induced norm. For instance the sequence $(u_n)_{n\in\mathbb{N}}$ with

$$u_n(x) = \begin{cases} 1, & \text{for } |x| \le 1 - 1/n, \\ 0, & \text{for } |x| \ge 1, \\ n(1-|x|), & \text{for } |x| \in (1-1/n, 1), \end{cases}$$

is a Cauchy sequence w.r.t. the norm induced by the L^2-scalar product. However, the limit function is not in C_b^0 although $u_n \in C_b^0$ for all $n \in \mathbb{N}$.

Since the limit of a Cauchy sequence of Riemann integrable functions is in general no longer Riemann integrable the Riemann integral has to be replaced by the Lebesgue integral in order to define complete function spaces [**Alt16**, §A1]. In order to define the Lebesgue and Sobolev spaces we introduce

$$C^\infty(\Omega,\mathbb{R}) = \{u : \Omega \to \mathbb{R} : u \text{ is arbitrarily many times differentiable}\}$$

and

$$C_0^\infty(\Omega,\mathbb{R}) = \{u \in C^\infty(\Omega,\mathbb{R}) : u \text{ has compact support in } \Omega\}$$

where the support of a function is defined by $\mathrm{supp}(u) = \mathrm{cl}_{\mathbb{R}^d}\{x \in \Omega : u(x) \neq 0\}$. The Lebesgue spaces are defined by

$$L^p(\Omega,\mathbb{R}) = \mathrm{cl}_{\|\cdot\|_{L^p}}(C_0^\infty(\Omega,\mathbb{R})), \quad \text{where} \quad \|u\|_{L^p} = \left(\int_\Omega |u(x)|^p\,\mathrm{d}x\right)^{1/p}$$

for all $p \in [1,\infty)$. By construction all these spaces are Banach spaces consisting of equivalence classes of Cauchy sequences, with two Cauchy sequences in the same class if their difference converges to zero. The L^p-spaces constructed in this way coincide with the spaces known from measure theory. The space $L^2(\Omega,\mathbb{R})$, respectively $L^2(\Omega,\mathbb{C})$, is a Hilbert space equipped with the scalar product (5.13).

For the solution of PDEs so called Sobolev spaces turn out to be useful. For $p\in[1,\infty)$ and Ω bounded we define

$$W^{m,p}(\Omega,\mathbb{R})=\mathrm{cl}_{\|\cdot\|_{W^{m,p}}}(C^\infty(\Omega,\mathbb{R})),$$

where

$$\|u\|_{W^{m,p}}=(\sum_{|j|\le m}\|\partial_x^j u\|_{L^p}^p)^{1/p}$$

and

$$W_0^{m,p}(\Omega,\mathbb{R})=\mathrm{cl}_{\|\cdot\|_{W^{m,p}}}(C_0^\infty(\Omega,\mathbb{R})),$$

for general Ω. Since the sum in the definition of $\|\cdot\|_{W^{m,p}}$ is finite there are various equivalent norms such as $\|u\|_{W^{m,p}}=\sum_{|j|\le m}\|\partial_x^j u\|_{L^p}$. By construction these spaces are Banach spaces, too. The spaces $H^m(\Omega,\mathbb{R})=W^{m,2}(\Omega,\mathbb{R})$ and $H_0^m(\Omega,\mathbb{R})=W_0^{m,2}(\Omega,\mathbb{R})$ are Hilbert spaces equipped with the scalar product

$$\langle u,v\rangle_{H^m}=\sum_{|j|\le m}\left\langle\partial_x^j u,\partial_x^j v\right\rangle_{L^2}.$$

By Sobolev's embedding theorem [**Alt16**, Satz 8.8.], Sobolev spaces can be embedded continuously into classical function spaces. We have

$$W^{m,p}(\Omega,\mathbb{R})\hookrightarrow C^{n,\alpha}(\overline{\Omega},\mathbb{R})\quad\text{if}\quad m-d/p>n+\alpha,$$

i.e., there exists a $C>0$, such that for all $u\in W^{m,p}$

$$\|u\|_{C^{n,\alpha}}\le C\|u\|_{W^{m,p}}$$

and in the equivalence class of $u\in W^{m,p}$ there is a representative $u\in C^{n,\alpha}(\overline{\Omega},\mathbb{R})$. For the proof of special cases see Lemma 5.1.27 and Lemma 5.2.3.

A different characterization of these spaces is (e.g., [**Alt16**, §1.25])

$$W^{m,p}(\Omega,\mathbb{R})=\{u:\Omega\to\mathbb{R}:\partial_x^\alpha u\in L^p\text{ for }|\alpha|=0,\dots,m,\ \|u\|_{W^{m,p}}<\infty\},$$

where $\partial_x^\alpha u$ denotes the α^{th} weak derivative of u. For $\Omega\subset\mathbb{R}^d$ the function $\partial_x^\alpha u\in L^p(\Omega,\mathbb{R})$ is called α^{th} weak derivative of $u\in L^p(\Omega,\mathbb{R})$ if for all $\phi\in C_0^\infty(\Omega,\mathbb{R})$ we have

$$\int_\Omega(\partial_x^\alpha u(x))\phi(x)\,\mathrm{d}x=(-1)^{|\alpha|}\int_\Omega u(x)(\partial_x^\alpha\phi(x))\,\mathrm{d}x.$$

We define $L^\infty(\Omega, \mathbb{R})$ as the space of all measurable functions $u : \Omega \to \mathbb{R}$ for which

$$\sup_{x \in \Omega \setminus N} |f(x)| < \infty \text{ for a null set } N.$$

This space is equipped with the norm

$$\|u\|_{L^\infty} = \inf_{N \text{ is a null set}} \sup_{x \in \Omega \setminus N} |f(x)| .$$

We introduce

$$\|u\|_{W^{m,\infty}} = \sum_{|\alpha|=0}^{m} \|\partial_x^\alpha u\|_{L^\infty}$$

and the space $W^{m,\infty}(\Omega, \mathbb{R})$ as the space of all functions $u : \Omega \to \mathbb{R}$ for which the weak derivatives $\partial_x^\alpha u$ exist for $|\alpha| = 0, \dots, m$ and for which $\|u\|_{W^{m,\infty}} < \infty$.

Remark 5.2.1. The concept of weak derivatives can be generalized to the concept of distributional derivatives [**RR04**, Chapter 5]. A priori, the sets C^∞ and C_0^∞ are just vector spaces. There is no norm for which these spaces are complete. However, using inductive limits of semi-norms, $C_0^\infty(\Omega, \mathbb{R})$ can be made to be a complete metric space $\mathcal{D}(\Omega)$, called space of test functions, where convergence $u_n \to u$ in $\mathcal{D}(\Omega)$ means: **a)** There exists a compact $K \subset \Omega$ such that $\operatorname{supp}(u_n), \operatorname{supp}(u) \subset K$, **b)** $\lim_{n\to\infty} \partial_x^\alpha u_n(x) = \partial_x^\alpha u(x)$ uniformly in K for all $\alpha \in \mathbb{N}^d$. However, this convergence is not induced by a norm.

The elements of the dual space of $\mathcal{D}(\Omega, \mathbb{R}) = C_0^\infty(\Omega, \mathbb{R})$ are called distributions, i.e., a distribution T is a continuous linear map from $\mathcal{D}$ into the real or complex numbers. This means that $u_n \to u$ in $\mathcal{D}$ implies $Tu_n \to Tu$, which is equivalent to the formulation that for all open bounded sets D there is a constant C and a number $m \in \mathbb{N}$ such that

$$|T(\phi)| \le C\|\phi\|_{C_b^m} \quad \text{for all} \quad \phi \in C_0^\infty(D, \mathbb{R}). \tag{5.14}$$

For a continuous function $u \in C_b^0(\mathbb{R}^d, \mathbb{R})$, or for u in one of the above other spaces

$$T_u(\phi) = \int_{\mathbb{R}^d} u(x)\phi(x)\,\mathrm{d}x$$

defines the so called associated distribution, which is then called regular. For the distribution associated to $\partial_x^\alpha u$ we find

$$\begin{aligned} T_{\partial_x^\alpha u}(\phi) &= \int_{\mathbb{R}^d} (\partial_x^\alpha u(x))\phi(x)\,\mathrm{d}x \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^d} u(x)(\partial_x^\alpha \phi(x))\,\mathrm{d}x = (-1)^{|\alpha|} T_u(\partial_x^\alpha \phi). \end{aligned}$$

This property is taken to define the α^{th} derivative of an arbitrary distribution by

$$(\partial_x^\alpha T)(\phi) = (-1)^{|\alpha|} T(\partial_x^\alpha \phi).$$

The distributional derivative of a function u is not necessarily again a function, as the next example shows. If $(\partial_x^\alpha T_u)$ can be represented by a function g, i.e., $\partial_x^\alpha T_u = T_g$, then g is the α^{th} weak derivative of u. ⌋

Example 5.2.2. For $u(x) = |x|$ we show that $u \in H^1((-1,1))$ but $u \notin H^2((-1,1))$ by computing the weak derivatives $\partial_x u, \partial_x^2 u$. For $\phi \in C_0^\infty((-1,1))$ we have $T_u(\partial_x \phi) = \int_{-1}^0 -x\partial_x\phi \,\mathrm{d}x + \int_0^1 x\partial_x\phi \,\mathrm{d}x = \int_{-1}^0 \phi \,\mathrm{d}x + \int_0^1 -\phi \,\mathrm{d}x = -T_g(\phi)$ with

$$g(x) = \partial_x u(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Similarly, $T_u(\partial_x^2 \phi) = \int_{-1}^0 \partial_x \phi(x) \,\mathrm{d}x + \int_0^1 -\partial_x\phi(x) \,\mathrm{d}x = 2\phi(0) = 2\delta_0(\phi)$ where δ_0 is called the Dirac δ distribution. Thus, $\partial_x^2 u \notin L^2$ as there is no function g such that $\int_{-1}^1 g(x)\phi(x) \,\mathrm{d}x = \phi(0)$. ⌋

In order to solve nonlinear PDEs two additional properties have to be satisfied by the function spaces in use. First, the values of the functions on the boundary have to be well defined. For $X = L^2(\Omega)$ a function $u \in L^2(\Omega)$ is only unique up to a null set in $\Omega \subset \mathbb{R}^d$. Since a smooth boundary $\partial\Omega$ is a null set, boundary conditions in L^2 are not well defined. Secondly, in the nonlinearity we have products of functions of X, i.e., with $u \in X$, also u^2 should be in X, i.e., X should be an algebra. For $u \in L^2$ in general we do not have $u^2 \in L^2$. However, for m sufficiently large (depending on p and the space dimension d), H^m or more generally $W^{m,p}$ are algebras, and point-wise values are defined, or at least the boundary conditions can be fulfilled in a generalized sense.

We close this subsection with the proof of a very simple version of Sobolev's embedding theorem and an example indicating which of the function spaces are suitable for solving PDEs and which are not.

Lemma 5.2.3. *Let $-\infty < a < b < \infty$. Then $H^1((a,b)) \subset C^{0,1/2}((a,b))$ and*

$$\|u\|_{L^\infty}^2 \leq 2\|u\|_{L^2} \left(\frac{1}{b-a} \|u\|_{L^2} + \|\partial_x u\|_{L^2} \right), \tag{5.15}$$

$$|u(x) - u(y)| \leq \sqrt{x-y} \|\partial_x u\|_{L^2}^2. \tag{5.16}$$

Proof. Since $C^1((a,b))$ is dense in $H^1((a,b))$ w.r.t. the $\|\cdot\|_{H^1}$-norm, it is sufficient to prove (5.15) and (5.16) for $u \in C^1((a,b))$. We have

$$\begin{aligned} u^2(x) &= \int_a^x \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s-a}{x-a}u^2(s)\right)\mathrm{d}s \\ &= \int_a^x \frac{1}{x-a}u^2(s)\,\mathrm{d}s + \int_a^x \frac{s-a}{x-a}2u(s)\partial_x u(s)\,\mathrm{d}s \\ &\le \frac{1}{x-a}\|u\|_{L^2}^2 + 2\|u\|_{L^2}\|\partial_x u\|_{L^2} \end{aligned}$$

and similarly

$$u^2(x) = \int_x^b \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s-x}{b-x}u^2(s)\right)\mathrm{d}s \le \frac{1}{b-x}\|u\|_{L^2}^2 + 2\|u\|_{L^2}\|\partial_x u\|_{L^2}.$$

Hence, $u^2(x) \le \min\{\frac{1}{x-a}, \frac{1}{b-x}\}\|u\|_{L^2}^2 + 2\|u\|_{L^2}\|\partial_x u\|_{L^2}$. For the second estimate we use the Cauchy-Schwarz inequality, namely

$$|u(x)-u(y)| = \left|\int_x^y \partial_s u(s)\,\mathrm{d}s\right| \le \int_x^y 1|\partial_s u(s)|\,\mathrm{d}s \le \sqrt{|x-y|}\|\partial_x u\|_{L^2}. \quad \square$$

We already stated that C^k-spaces are in general not optimal concerning the regularity of solutions of PDEs. More life is given to this statement by the following example [**Sal08**, Example 8.2]. This gives a motivation for the use of Sobolev spaces, in particular for the use of H^m-spaces for which Hilbert space methods are available.

Example 5.2.4. For $0 < \alpha < 2\pi$ let $\Omega_\alpha := \{(r,\phi) : 0 < r < 1, -\alpha/2 < \theta < \alpha/2\}$ be the two-dimensional sector with opening angle α. Consider the Dirichlet boundary value problem

$$-\Delta u = 0 \text{ in } \Omega_\alpha, \quad u|_{\partial\Omega} = g_\alpha(r,\phi) \text{ on } \partial\Omega_\alpha, \tag{5.17}$$

with $g_\alpha(r,\phi) = \cos(\pi\phi/\alpha)$ for $r = 1$, $g(r,\phi) = 0$ else, where (r,ϕ) are polar coordinates. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ we find that $f(z) = z^{\pi/\alpha}$ is holomorphic in Ω_α, and thus

$$u(r,\phi) = \mathrm{Re}(f(z)) = r^{\pi/\alpha}\cos(\pi\phi/\alpha)$$

is harmonic in Ω_α and satisfies the boundary conditions. Thus, it is the unique solution of (5.17).

Clearly, $u \in C^\infty(\Omega)$, and we now consider the regularity of u up to the boundary and compare it with Sobolev regularity. Let $\alpha \ne \pi$, otherwise $u(x,y) = x$. We find

$$|\nabla u|^2 = (\partial_r u)^2 + \frac{1}{r^2}(\partial_\phi u)^2 = \frac{\pi^2}{\alpha^2}r^{2(\pi/\alpha-1)},$$

and this is in $C^1(\overline{\Omega})$ only for $\alpha \le \pi$. But $\int_{\Omega_\alpha}|\nabla u|^2\,\mathrm{d}x = \frac{\pi^2}{\alpha}\int_0^1 r^{2\pi/\alpha-1}\,\mathrm{d}r = \pi/2$ independent of α, and thus $u \in H^1(\Omega)$ for all α. Next, $|\partial_{x_i}^2 u| \sim$

$r^{\pi/\alpha-2}$ for $r \to 0$, thus, $u \in C^2(\overline{\Omega})$ for $\alpha \le \pi/2$, and $u \in H^2(\Omega)$ for $\alpha \le \pi$, i.e., if the sector is convex. By setting $\widetilde{g}(r,\phi) = r^j g(\phi)$ with $j \ge 2$ and $v = u - \widetilde{g}$ we find $\widetilde{g} \in C^\infty(\overline{\Omega_\alpha})$ and $-\Delta v = -\Delta u + \Delta\widetilde{g} = \Delta\widetilde{g} =: f$ in Ω_α, and $v|_{\partial\Omega_\alpha} = 0$. For the last system the Lax-Milgram theorem [**Eva98**, §6.2.1] guarantees $u \in H^1$ for general spatial domains and regularity theory [**Eva98**, §6.3] guarantees $u \in H^2$ for convex spatial domains. ⌋

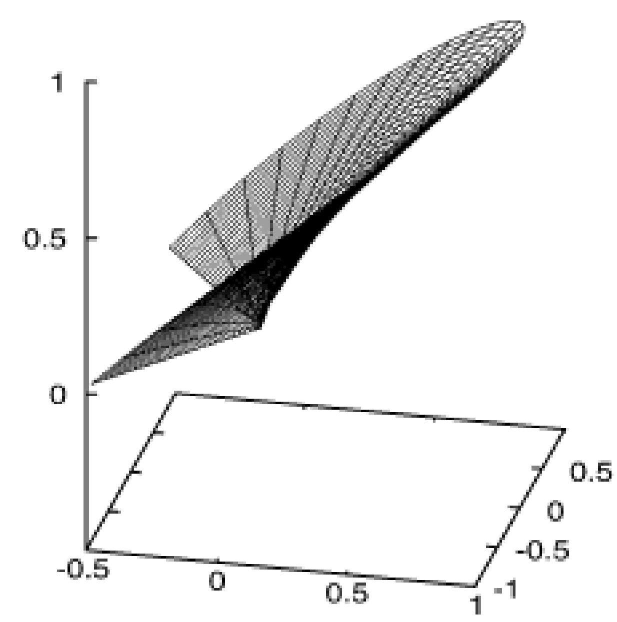


Figure 5.3. $z \mapsto (\mathrm{Re} z)^{3/4}$ solves the boundary value problem (5.17) if $\alpha = 4\pi/3$. The derivative is unbounded at the origin.

5.2.2. Fourier series. PDEs with periodic boundary conditions for the spatial coordinates can be transferred to ODEs in $\mathbb{R}^\infty$ with the help of Fourier series. For notational simplicity we restrict ourselves first to the torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. Let C^∞_{per} be the space of functions $u : \mathbb{T}^d \to \mathbb{R}^d$, with a C^∞ periodic extension $\widetilde{u} : \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\begin{aligned}\widetilde{u}(x_1, x_2, \ldots, x_d) &= \widetilde{u}(x_1 + 2\pi, x_2, \ldots, x_d) = \widetilde{u}(x_1, x_2 + 2\pi, \ldots, x_d) \\ &= \ldots = \widetilde{u}(x_1, x_2, \ldots, x_d + 2\pi).\end{aligned}$$

We define

$$H^m_{\text{per}} = \mathrm{clos}_{\|\cdot\|_{H^m(\mathbb{T}^d)}}(C^\infty_{\text{per}}).$$

The question is if and in what sense a function can be represented by its Fourier series, or equivalently, in which norms Fourier series converge. In L^2 we have a simple answer which follows from the general theory of orthonormal systems, and which for convenience we summarize here.

Definition 5.2.5. *Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$. A (finite or infinite) system (ϕ_j) in H, $j = 1, \ldots, N$ or $j \in \mathbb{N}$, is called orthogonal system if $\langle\phi_i, \phi_j\rangle = 0$ for $i \ne j$. It is called orthonormal if additionally $\langle\phi_j, \phi_j\rangle = 1$. It is called a complete orthonormal system (complete ONS) or Hilbert basis if $\langle u, \phi_j\rangle = 0$ for all j implies $u = 0$ for $u \in H$.*

In Hilbert spaces H the following holds.

Lemma 5.2.6. *a) If $u_n \to u$ and $v_n \to v$ in H, then $\langle u_n, v_n\rangle \to \langle u, v\rangle$.*
b) (Pythargoras) For $\phi_1, \ldots, \phi_n \in H$ with $\langle \phi_i, \phi_j\rangle = 0$ for $i \neq j$ we have

$$\|\phi_1 + \ldots + \phi_n\|^2 = \|\phi_1\|^2 + \ldots + \|\phi_n\|^2.$$

c) If $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal system in H, and u_j a sequence in $\mathbb{C}$, then $\sum_{j=1}^{\infty} u_j\phi_j$ converges if and only if $\sum_{j=1}^{\infty} |u_j|^2$ converges, i.e., if $(u_j)_{j\in\mathbb{N}} \in \ell^2$.
d) (Bessel's inequality) If $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal sequence in H, then for all $u \in H$ we have $\sum_{j=1}^{\infty} |\langle u, \phi_j\rangle|^2 \le \|u\|^2$.

Proof. a) By Cauchy-Schwarz we have

$$\begin{aligned} |\langle u_n, v_n\rangle - \langle u, v\rangle| &= |\langle u_n, v_n - v\rangle + \langle u_n - u, v\rangle| \\ &\le \|u_n\|\|v_n - v\| + \|u_n - u\|\|v\|, \end{aligned}$$

where $\sup_{n\in\mathbb{N}} \|u_n\| < \infty$ for convergent series $(u_n)_{n\in\mathbb{N}}$.
b) Direct calculation for the finite sums.
c) For all $m \le n$ we have $\|\sum_{j=m}^{n} u_j\phi_j\|^2 = \sum_{j=m}^{n} |u_j|^2$ by b). Thus, $\sum_{j=1}^{\infty} u_j\phi_j$ is a Cauchy sequence if and only if $\sum_{j=1}^{\infty} |u_j|^2$ is a Cauchy sequence.
d) For $N \in \mathbb{N}$ we have

$$0 \le \left\langle u - \sum_{j=1}^{N} \langle u, \phi_j\rangle \phi_j, u - \sum_{j=1}^{N} \langle u, \phi_j\rangle \phi_j \right\rangle = \|u\|^2 - \sum_{j=1}^{N} |\langle u, \phi_j\rangle|^2,$$

and hence $\sum_{j=1}^{N} |\langle u, \phi_j\rangle|^2 \le \|u\|^2$, which implies convergence of the series and Bessel's inequality. □

Lemma 5.2.7. *The following statements are equivalent:*
(i) $(\phi_j)_{j\in\mathbb{N}}$ is a complete ONS.
(ii) For all $u \in H$ we have $u = \sum_{j=1}^{\infty} \langle u, \phi_j\rangle \phi_j$.
(iii) For all $u, v \in H$ we have Parseval's identity

$$\langle u, v\rangle = \sum_{j=1}^{\infty} \langle u, \phi_j\rangle \langle v, \phi_j\rangle . \tag{5.18}$$

(iv) For all $u \in H$ we have Bessel's equality

$$\|u\|^2 = \sum_{j=1}^{\infty} |\langle u, \phi_j\rangle|^2. \tag{5.19}$$

Proof. (i)⇒(ii). For $u \in H$ we have convergence of $\sum_{j=1}^{\infty} |\langle u, \phi_j\rangle|^2$ by d) and convergence of $\sum_{j=1}^{\infty} \langle u, \phi_j\rangle \phi_j$ to some $v \in H$ by Lemma 5.2.6 c). By Lemma 5.2.6 a) we have

$$\langle u - v, \phi_j\rangle = \langle u, \phi_j\rangle - \sum_{n=1}^{\infty} \langle u, \phi_n\rangle \langle \phi_n, \phi_j\rangle = \langle u, \phi_j\rangle - \langle u, \phi_j\rangle = 0,$$

and since (ϕ_j) is complete this implies $v = u$. (ii)$\Rightarrow$(iii) again follows from Lemma 5.2.6 a), and (iv) follows from (iii) with $v = u$. Finally, (iv)$\Rightarrow$(i) since $\langle u, \phi_j\rangle = 0$ for all j and (iv) imply $\|u\| = 0$, hence $u = 0$. $\square$

Due to the equivalence of (5.18) and (5.19), often both are called Parseval's identity. Clearly, Lemma 5.2.6 and Lemma 5.2.7 also holds if sequences $(\phi_j)_{j\in\mathbb{N}}$ are replaced by sequences $(\phi_j)_{j\in\mathbb{Z}}$, $(\phi_j)_{j\in\mathbb{N}^d}$, $(\phi_j)_{j\in\mathbb{Z}^d}$, with the respective replacements in the sums. The most important example are classical Fourier series.

Theorem 5.2.8. *a) The functions $\phi_k = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}k\cdot x}$ with $k \in \mathbb{Z}^d$ are a complete ONS in $L^2(\mathbb{T}^d)$ w.r.t. the inner product*

$$\langle u, v\rangle_{L^2} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} u(x)\overline{v(x)}\,\mathrm{d}x.$$

For $u \in L^2(\mathbb{T}^d)$ we have L^2-convergence of the Fourier series, i.e., for

$$S_N(x) = \sum_{|k|\le N} \widehat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \qquad \text{with} \quad \widehat{u}_k = \langle \mathrm{e}^{\mathrm{i}k\cdot x}, u\rangle_{L^2} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} u(x)\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x$$

we have $\|u - S_N\|_{L^2} \to 0$ as $N \to \infty$. This convergence is abbreviated as $u(x) = \sum_{k\in\mathbb{Z}^d} \widehat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}$.

b) For all $\phi \in T_N = \operatorname{span}\{\mathrm{e}^{\mathrm{i}k\cdot x} : |k| \le N\}$ we have $\|u - S_N\|_{L^2} \le \|u - \phi\|_{L^2}$, i.e., S_N is the best approximation of u in T_N in the quadratic mean.

c) We have Parseval's identity

$$\sum_{k\in\mathbb{Z}^d} |\widehat{u}_k|^2 = \frac{1}{(2\pi)^d}\|u\|_{L^2}^2.$$

d) There exists a $C > 0$ such that if $u \in C^m$ is 2π-periodic in each direction, then $|\widehat{u}_k| \le C|k|^{-m}$.

Proof. a) By direct calculation we find $\langle \phi_k, \phi_m\rangle = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{else,} \end{cases}$ i.e., the (ϕ_k) are an ONS. The completeness of this ONS can be shown with the Weierstraß approximation theorem, see [**Alt16**, Satz 7.10].

b) follows since S_N is the orthogonal projection of u on T_N.

c) Parseval's identity can be computed directly for finite sums. Going to the limits shows the assertion.

d) Through integration by parts we find

$$\begin{aligned}\widehat{u}_k = \langle \mathrm{e}^{\mathrm{i}k\cdot x}, u\rangle_{L^2} &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} u(x)\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x \\ &= (-1)^n \frac{1}{(2\pi)^d k^m}\int_{\mathbb{T}^d} (\partial_x^m u(x))\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x.\end{aligned}$$

□

The map $u \mapsto (\widehat{u})_{k\in\mathbb{Z}^d}$ will be abbreviated with $\mathcal{F}$. By c), $\mathcal{F}$ is an isometric isomorphism from L^2 to ℓ_2. Its inverse $(\widehat{u}_k)_{k\in\mathbb{Z}} \mapsto u$ is denoted by $\mathcal{F}^{-1}$. By d) the smoothness of u is related to the decay of its Fourier coefficients.

Formally we have $\partial_x u(x) = \sum_{k\in\mathbb{Z}} \mathrm{i}k\widehat{u}_k \mathrm{e}^{\mathrm{i}kx}$, or equivalently $\mathcal{F}(\partial_x u) = (\mathrm{i}k\widehat{u}_k)_{k\in\mathbb{Z}}$. It follows, that $\mathcal{F}$ is in fact an isomorphism between the Sobolev spaces H^m_{per} and the spaces of sequences $\ell_{2,m}$ which have been introduced in §5.1. Moreover, $\mathcal{F}^{-1}$ maps $\ell_{1,m}$ to C^m_b.

Lemma 5.2.9. *Let $m \in \mathbb{N}_0$. a) There exists a $C > 0$, such that for all $\widehat{u} \in \ell_{1,m}$*

$$\|u\|_{C^m_b} \leq C\|\widehat{u}\|_{\ell_{1,m}}.$$

b) There exist $C_1, C_2 > 0$, such that for all $\widehat{u} \in \ell_{2,m}$

$$C_1\|\widehat{u}\|_{\ell_{2,m}} \leq \|u\|_{H^m} \leq C_2\|\widehat{u}\|_{\ell_{2,m}}.$$

c) There exist $C_1, C_2 > 0$, such that for all $u \in H^m_{per}$

$$C_1\|u\|_{H^m} \leq \|\widehat{u}\|_{\ell_{2,m}} \leq C_2\|u\|_{H^m}.$$

Proof. For notational simplicity we consider $d = 1$ and u with $\widehat{u}_0 = 0$. Moreover, we first consider $\mathcal{F}$ and $\mathcal{F}^{-1}$ on the dense subspaces C^∞_{per} respectively the space of finite sequences. The results then follow by continuous extension, see the subsequent Lemma 5.2.10.

a) We have

$$\begin{aligned}\|u\|_{C^m_b} &\leq C\sup_{x\in\mathbb{R}}\sup_{0\leq j\leq m}\left|\partial_x^j \sum_{k\in\mathbb{Z}} \widehat{u}_k \mathrm{e}^{\mathrm{i}kx}\right| \leq C\sup_{x\in\mathbb{R}}\sup_{0\leq j\leq m}\sum_{k\in\mathbb{Z}} |k|^j|\widehat{u}_k|\ |\mathrm{e}^{\mathrm{i}kx}| \\ &\leq C\|\widehat{u}\|_{\ell_{1,m}}.\end{aligned}$$

Continuity in respect to the differentiability of u follows from the uniform and absolute convergence of the series.

b) and c) The second estimate in b) and the first estimate in c) follow from

$$\begin{aligned}
\|u\|_{H^m}^2 &= \sum_{j=0}^m \|\partial_x^j u\|_{L^2}^2 = \sum_{j=0}^m \int_{\mathbb{T}^d} \left| \partial_x^j \sum_{k\in\mathbb{Z}} \widehat{u}_k \mathrm{e}^{ikx} \right|^2 dx \\
&= \sum_{j=0}^m \int_{\mathbb{T}^d} \left| \sum_{k\in\mathbb{Z}} \widehat{u}_k (ik)^j \mathrm{e}^{ikx} \right|^2 dx \le 2\pi \sum_{j=0}^m \sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 |k|^{2j} \le C\|\widehat{u}\|_{\ell_{2,m}}^2.
\end{aligned}$$

The first estimate in b) and the second estimate in c) follow from

$$\begin{aligned}
\|\widehat{u}\|_{\ell_{2,m}}^2 &= \sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 |k|^{2m} = \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{T}^d} \mathrm{e}^{-ikx} u(x)dx \right|^2 |k|^{2m} \\
&\overset{\int \text{by parts}}{=} \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{T}^d} \mathrm{e}^{-ikx} \partial_x u(x)dx \right|^2 |k|^{2(m-1)} = \dots \\
&= \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{T}^d} \mathrm{e}^{-ikx} \partial_x^m u(x)dx \right|^2 \\
&= \|\mathcal{F}(\partial_x^m u)\|_{\ell_{2,0}}^2 \overset{\text{Parseval}}{=} \frac{1}{2\pi} \|\partial_x^m u\|_{L^2}^2 \le \|u\|_{H^m}^2.
\end{aligned}$$

□

Lemma 5.2.10. *Let X be a metric space, $A \subset X$ a dense set, and Y a complete metric space. Then every uniformly continuous function $f : A \to Y$ possesses a unique uniformly continuous extension $\widetilde{f} : X \to Y$.*

Proof. The condition that the extension must be continuous leads to the only possible extension of f, namely

$$\widetilde{f}(x) = \lim_{x'\in A, x'\to x} f(x').$$

It remains to prove the existence of this limit, i.e., to prove that $\widetilde{f}$ is well defined. In order to do so, let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in A$ and $\lim_{n\to\infty} x_n = x$. Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and from the uniform continuity it follows that the image sequence $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since Y is complete, we have the existence of

$$y = \lim_{n\to\infty} f(x_n)$$

in Y. Obviously the limit y is independent of the chosen sequence. It is an easy exercise to prove the uniform continuity of $\widetilde{f}$. □

We give a number of remarks and further results about Fourier series which will be useful later.

Remark 5.2.11. (Hausdorff-Young) ßThe discrete Fourier transform $\mathcal{F}$ is continuous from L^p to ℓ_q with $1/p+1/q=1$ for $p \leq 2$. The discrete inverse Fourier transform $\mathcal{F}^{-1}$ is continuous from ℓ_q to L^p with $1/p+1/q=1$ for $q \leq 2$, but not for $q > 2$, cf. [**Duo01**, Corollary 1.20]. This can be shown with the so called Riesz-Thorin interpolation between the inequalities from Theorem 5.2.8 c) and Lemma 5.2.9 a).

Remark 5.2.12. Lemma 5.2.9 suggests to define non-integer Sobolev spaces by Fourier series, i.e., for $\theta \in \mathbb{R}$ let

$$H^\theta_{per} = \mathcal{F}^{-1}\ell_{2,\theta} \quad \text{with} \quad \|u\|_{H^\theta} = \|\widehat{u}\|_{\ell_{2,\theta}}.$$

We will come back to this definition in §6.2.1. ⌋

Remark 5.2.13. (Real Fourier series) Besides the complex Fourier expansion also real Fourier polynomials and series of the form

$$u(x) = \frac{a_0}{2} + \sum_{k\in\mathbb{N}^d}^{\infty} [a_k \cos(k\cdot x) + b_k \sin(k\cdot x)],$$

with $a_k, b_k \in \mathbb{R}$, are in use, where

$$\begin{aligned} a_k =& \frac{2}{(2\pi)^d}\int_{\mathbb{T}^d} u(x)\cos(k\cdot x)\,\mathrm{d}x, \quad k \geq 0,\\ b_k =& \frac{2}{(2\pi)^d}\int_{\mathbb{T}^d} u(x)\sin(k\cdot x)\,\mathrm{d}x, \quad k \geq 1. \end{aligned}$$

The relations between $\widehat{u}_k$ and (a_k, b_k) are

$$\widehat{u}_0 = \frac{1}{2}a_0, \quad \widehat{u}_k = \frac{1}{2}(a_k - \mathrm{i}b_k), \quad \widehat{u}_{-k} = \frac{1}{2}(a_k + \mathrm{i}b_k),$$

$$a_k = \widehat{u}_k + \widehat{u}_{-k}, \quad b_k = (\widehat{u}_k - \widehat{u}_{-k})\mathrm{i}, \quad k \in \mathbb{N}.$$

For $u(x) \in \mathbb{R}$ we have $\widehat{u}_k = \overline{\widehat{u}}_{-k}$. In this book we prefer the concise complex notation. ⌋

Remark 5.2.14. (General periodic boxes) Let $L_1, \ldots, L_d > 0$, $\Omega = (0, L_1) \times \cdots \times (0, L_d)$. As in Theorem 5.2.8 we may expand $u \in L^2(\Omega)$ as

$$u(x) = \sum_{k\in\mathbb{Z}^d}^{\infty} \widehat{u}_k \mathrm{e}^{\mathrm{i}\omega_k\cdot x} \quad \text{with} \quad \omega_k = \left(\frac{2\pi k_1}{L_1}, \ldots, \frac{2\pi k_d}{L_d}\right),$$

where

$$\widehat{u}_k = \frac{1}{L_1 L_2 \cdots L_d}\int_\Omega u(x)\mathrm{e}^{-\mathrm{i}\omega_k\cdot x}\,\mathrm{d}x = (\mathcal{F}u)_k.$$

Again, $\mathcal{F}$ is an isomorphism between $H^m_{\mathrm{per}}(\Omega)$ and $\ell_{2,m}$. ⌋

Remark 5.2.15. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $(\varphi_j)_{j\in\mathbb{N}}$ be a complete orthonormal system in $L^2(\Omega)$, i.e., every $u \in L^2(\Omega)$ possesses a unique representation as convergent series in $L^2(\Omega)$, i.e., $u = \sum_{j\in\mathbb{N}} c_j\varphi_j$ with $c_j \in \mathbb{C}$. Then $\mathcal{F}$, defined by $(\mathcal{F}u)_j = c_j$, is an isomorphism between $L^2(\Omega)$ and ℓ_2. However, in general $\mathcal{F}$ is not an isomorphism between $H^m(\Omega)$ and $\ell_{2,m}$. The set $\{\sin nx : n \in \mathbb{N}\}$ is a basis of $L^2((0,\pi))$, but not a basis of $H^1((0,\pi))$. Since $H^1 \subset C_b^0$ in H^1, only functions u with $u(0) = u(\pi) = 0$ can be approximated. In L^2 the two points $x = 0, \pi$ are a null set. ⌋

Remark 5.2.16. Point-wise convergence of Fourier series is a rather delicate issue. For instance, the Fourier series of $u \in L^1$ may diverge almost everywhere [**Kol27**], while for $u \in L^2$ we have convergence almost everywhere [**Car66**]. No necessary and sufficient conditions are known for the point-wise convergence of the Fourier series of a function u. However, there are various sufficient conditions, for instance if u is piecewise C^1, then $S_n(x) \to u(x)$ at points of continuity. More generally,

$$S_n(x) \to \frac{1}{2}(u(x+) + u(x-)),$$

where $u(x+)$ and $u(x-)$ denote the right and the left limit of u in x. Thus, $S_n(x_0)$ converges to the mean of u at jump points x_0. This convergence comes with notable oscillations ($\approx 19\%$) to the left and right of x_0, which is known as Gibbs phenomenon, see Figure 5.4. ⌋

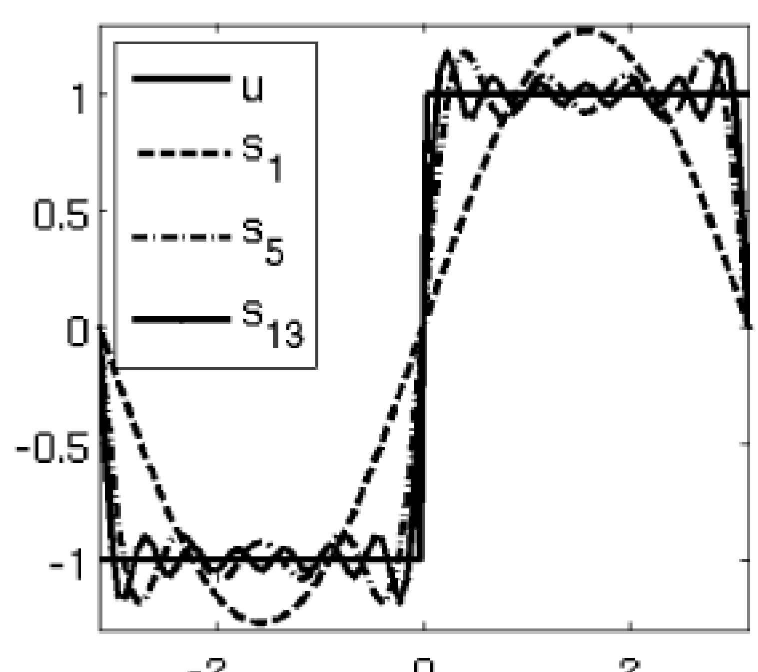


Figure 5.4. Fourier expansion of $u(x) = -1$, $x \in (-\pi, 0)$, $u(x) = 1$, $x \in (0,\pi)$ yields $u(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$. The figure shows the partial sums s_1, s_5, s_{13} and thus illustrates the Gibbs phenomenon.

Fourier series allow giving simple proofs of classical inequalities when the functions involved are spatially periodic.

Lemma 5.2.17. (Poincaré's inequality) *For* $u \in H^1_{\text{per}}(\mathbb{T}^d, \mathbb{R})$ *with* $\int_{\mathbb{T}^d} u \,\mathrm{d}x = 0$ *we have*

$$\int_{\mathbb{T}^d} |u|^2 \,\mathrm{d}x \le \int_{\mathbb{T}^d} |\nabla u|^2 \,\mathrm{d}x. \tag{5.20}$$

Proof. Parseval's identity gives

$$\int_{\mathbb{T}^d} |u|^2 \,\mathrm{d}x = 2\pi \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\widehat{u}_k|^2 \le 2\pi \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^2 |\widehat{u}_k|^2 = \int_{\mathbb{T}^d} |\nabla u|^2 \,\mathrm{d}x. \quad \square$$

Next we have the following version of Sobolev's embedding theorem in d space dimensions.

Lemma 5.2.18. *For $m - d/2 > n$ there exists a $C > 0$, such that*

$$\|u\|_{C^n(\mathbb{T}^d)} \le C \|u\|_{H^m(\mathbb{T}^d)}.$$

Proof. The assertion follows from Lemma 5.2.9 and Lemma 5.1.27. $\square$

Analytic properties of the solution operator of a linear evolution equation can be established with the help of Fourier series.

Example 5.2.19. We consider the solution operator $T(t)$ defined via the solution $u(x,t) = T(t)u_0)(x)$ of the linear heat equation $\partial_t u = \partial_x^2 u$, with $x \in [0,\pi]$, under Dirichlet boundary condition $u(0,t) = u(\pi,t) = 0$ to the initial value $u(x,0) = u_0(x)$. In order to prove that $(T(t))_{t\ge 0}$ is a C_0-semigroup in $L^2((0,\pi))$ and in $H^m((0,\pi)) \cap H_0^1((0,\pi))$ for every $m \in \mathbb{N}$ we make an odd 2π-periodic extension of the functions with $u(0,t) = u(\pi,t) = 0$. The semigroup in the space of 2π-periodic functions is denoted again by $T(t)$. We proved in §5.1.2 that $\widehat{T}(t) = \mathcal{F}T(t)\mathcal{F}^{-1}$ defined by $(\widehat{T}(t)\widehat{u}(0))_{k\in\mathbb{Z}} = (\mathrm{e}^{-k^2 t}\widehat{u}_k(0))_{k\in\mathbb{Z}}$ is continuous in $\ell_{2,m}$, i.e., for every $\widehat{u}(0) \in \ell_{2,m}$ we have

$$\|\widehat{T}(t)\widehat{u}(0) - \widehat{u}(0)\|_{\ell_{2,m}} \to 0 \quad \text{for} \quad t \to 0 \,.$$

Due to the isomorphism property of $\mathcal{F}$ between H^m_{per} and $\ell_{2,m}$, cf. Lemma 5.2.9 b) and c), it follows that

$$\begin{aligned} \|T(t)u(0) - u(0)\|_{H^m_{per}} &\le C_1 \|\mathcal{F}^{-1}T(t)u(0) - \widehat{u}(0)\|_{\ell_{2,m}} \\ &= C_1 \|\widehat{T}(t)\widehat{u}(0) - \widehat{u}(0)\|_{\ell_{2,m}} \to 0 \ \text{ for } \ t \to 0 \,, \end{aligned}$$

i.e., $T(t)$ is a C_0-semigroup in H^m_{per}. The restriction of x to $[0,\pi]$ gives the result. Moreover, from Example 5.1.21 it is known that for $r \ge 0$ the semigroup $\widehat{T}(t)$ can be estimated by

$$\|\widehat{T}(t)\widehat{u}(0)\|_{\ell_{2,m+r}} \le \widetilde{C} \max(1, t^{-r/2}) \|\widehat{u}(0)\|_{\ell_{2,m}} \,.$$

Using again that $\mathcal{F}$ is an isomorphism between H^m_{per} and $\ell_{2,m}$ shows

$$\begin{aligned} \|T(t)u_0\|_{H^{m+r}_{per}} &\le C_1 \|\widehat{T}(t)\widehat{u}(0)\|_{\ell_{2,m+r}} \le C_1 \widetilde{C} \max(1, t^{-r/2}) \|\widehat{u}(0)\|_{\ell_{2,m}} \\ &\le C \max(1, t^{-r/2}) \|u_0\|_{H^m_{per}} \,, \end{aligned}$$

with $C = C_1 \widetilde{C} C_2$. ⌋

Example 5.2.20. We consider $\partial_t^2 u = \partial_x^2 u$ with $x \in (0, 2\pi)$ and periodic boundary conditions. We rewrite this equation as first order system for $z = (\partial_t u, \partial_x u)$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} z(x,t) = A z(x,t), \quad \text{with} \quad A = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix},$$

or in Fourier space

$$\frac{\mathrm{d}}{\mathrm{d}t} \widehat{z}(k,t) = \widehat{A} \widehat{z}(k,t), \quad \text{with} \quad \widehat{A} = \begin{pmatrix} 0 & \mathrm{i}k \\ \mathrm{i}k & 0 \end{pmatrix}.$$

The general solution is given by

$$z(x,t) = \sum_{k \in \mathbb{Z}} c_1(k) \mathrm{e}^{\mathrm{i}k(t+x)} \widehat{z}_1 + c_2(k) \mathrm{e}^{\mathrm{i}k(-t+x)} \widehat{z}_2 =: \mathrm{e}^{tA} z(\cdot, 0)(x),$$

where

$$\begin{pmatrix} c_1(k) \\ c_2(k) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \widehat{z}(k, 0).$$

Hence, we have a uniformly bounded C_0-semigroup for z in, e.g., $H^m \times H^m$, which however is not smoothing. ⌋

5.2.3. Some nonlinear PDE examples. We start this section with a version of the local existence and uniqueness theorem, Theorem 5.1.23, in physical space. In order to use the results from §5.1 we refrain from greatest generality and restrict ourselves to Sobolev spaces H^θ_{per} in accordance with Lemma 5.2.9. We consider

$$\frac{\mathrm{d}}{\mathrm{d}t} u = \Lambda u + N(u), \quad u|_{t=0} = u_0, \tag{5.21}$$

with Λ generating a C_0-semigroup which satisfies

$$\|\mathrm{e}^{t\Lambda} u\|_{H^\theta_{per}} \leq C_{\theta - r} \mathrm{e}^{\beta t} t^{-\alpha} \|u\|_{H^r_{per}} \tag{5.22}$$

for $\theta \geq r$ with constants $C_{\theta-r}$, β, and $\alpha \in [0, 1)$. Moreover, let N be a locally Lipschitz-continuous map from H^θ into H^r.

Similar to Definition 5.1.19 we define

Definition 5.2.21. *a) A function $u \in C([0, T_0], H^\theta_{per})$ which satisfies*

$$u(t) = \mathrm{e}^{t\Lambda} u_0 + \int_0^t \mathrm{e}^{(t-\tau)\Lambda} N(u(\tau)) \, \mathrm{d}\tau \tag{5.23}$$

is called a mild solution of (5.21).
b) A function $u \in C^1([0, T_0], H^\theta_{per})$, with $\Lambda u \in C([0, T_0], H^\theta_{per})$, is called a strong solution of (5.21), if (5.21) holds in H^θ_{per} for each $t \in [0, T_0]$.

Theorem 5.2.22. *For all $C_1 > 0$ there exists a $T_0 > 0$ such the following holds. For $u_0 \in H^\theta$ with $\|u_0\|_{H^\theta} \leq C_1$ there exists a unique solution $u \in C([0, T_0], H^\theta)$ of (5.21) with $u|_{t=0} = u_0$.*

Proof. This a direct consequence of Theorem 5.1.23, Remark 5.1.24, and Lemma 5.2.9. □

There are other straightforward generalizations from the finite- to the infinite-dimensional situation.

Theorem 5.2.23. *Consider* (5.21)*, where* (5.22) *is satisfied for a* $\beta < 0$. *Then the fixed point* $u^* = 0$ *is asymptotically stable.*

Proof. The proof goes line for line as the proof of Theorem 2.3.4 a). □

In order to check the assumptions for system (5.21) for a concrete nonlinear PDE we have to handle products of functions in physical space. The H^θ_{per}-spaces are closed under multiplication if θ is sufficiently big.

Lemma 5.2.24. *For all* $\theta > d/2$ *there exists a* $C > 0$*, such that for all* $u, v \in H^\theta_{per}$ *we have*

$$\|uv\|_{H^\theta} \leq C\|u\|_{H^\theta}\|v\|_{H^\theta}.$$

Proof. This follows from Lemma 5.1.28 by using the isomorphism $\mathcal{F} : H^\theta_{per} \to \ell_{2,\theta}$. □

We come back to the PDEs introduced in Example 5.1.29. We have already proved the local existence and uniqueness of solutions of the Fourier transformed versions. The isomorphism property between $\ell_{2,\theta}$ and H^θ_{per} gives the following result.

Theorem 5.2.25. *For the KPP equation, the Allen-Cahn equation, the NLS equation, the Burgers equation, and the GL equation with* 2π*-periodic boundary conditions we have the local existence and uniqueness of solutions in* H^θ_{per} *if* $\theta > 1/2$*, i.e., for all* $C > 0$ *there exists a* $T_0 > 0$ *such the following holds. For* $u_0 \in H^\theta_{per}$ *with* $\|u_0\|_{H^\theta} \leq C$ *there exists a unique solution* $u \in C([0, T_0], H^\theta_{per})$ *with* $u|_{t=0} = u_0$*, respectively,* $A \in C([0, T_0], H^\theta_{per})$ *with* $A|_{T=0} = A_0$.

The θ in the last theorem can be made smaller by using the smoothing properties of the semigroup, cf. §6.2.1. Moreover, the smoothing estimate (5.11) can be used to show that solutions to the KPP equation, the Allen-Cahn equation, and the GL equation become arbitrary smooth and even analytic for $t > 0$. This is done for instance in §5.3.3 or §6.2.2.

5.3. The Chafee-Infante problem

After having discussed the local existence and uniqueness theory of PDEs on an interval we now consider the qualitative behavior of solutions in a specific example, namely the Chafee-Infante problem [**CI75**]. The presentation is based on [**Hen81**, §5.3].

The Chafee-Infante problem is to find the attractor of a semi-linear parabolic PDE, the Allen-Cahn equation,

$$\partial_t u = \partial_x^2 u + \alpha u - u^3, \tag{5.24}$$

with $\alpha \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, $t \geq 0$, and $x \in (0,\pi)$, under Dirichlet boundary conditions $u(0,t) = u(\pi,t) = 0$. Our goal is to characterize the attractor of this system for different values of α. This PDE can be interpreted as an infinite-dimensional gradient system. Similar to finite-dimensional gradient systems, see §2.4.5, this fact restricts the elements of the attractor in the following to fixed points and heteroclinic connections.

5.3.1. Local and global existence of solutions. As phase space we use $H_0^1 = H_0^1(0,\pi)$. Solving the Allen-Cahn equation with 2π-periodic boundary conditions and restricting to the invariant subspace of odd functions is the same as solving the Allen-Cahn equation with Dirichlet boundary conditions. Hence, Theorem 5.2.25 applies and we have the local existence and uniqueness of solutions in H^θ if $\theta > 1/2$.

Theorem 5.3.1. *For all $C > 0$ there exists a $T_0 > 0$ such that for all $u_0 \in H_0^1$ with $\|u_0\|_{H^1} \leq C$ there exists a unique solution $u \in C([0,T_0], H_0^1)$ of the Allen-Cahn equation* (5.24) *with $u|_{t=0} = u_0$.*

To prove the global existence of solutions it is sufficient to bound the H^1-norm. We prove more, namely the existence of an absorbing set for (5.24).

Theorem 5.3.2. (Global existence and existence of an absorbing set) *For all $\alpha \in \mathbb{R}$ there exists a $R > 0$ such that for all $C_1 \geq 0$ we have a $T > 0$ such that the followings holds. If $u_0 \in H_0^1$ satisfies $\|u_0\|_{H^1} \leq C_1$, then the associated solution satisfies $u(t) \in B = \{u \in H_0^1 : \|u\|_{H^1} \leq R\}$ for all $t \geq T$.*

Proof. Again the solutions are extended to odd 2π-spatially periodic solutions. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} u^2 \,\mathrm{d}x = 2\int_0^{2\pi} -(\partial_x u)^2 + \alpha u^2 - u^4 \,\mathrm{d}x \tag{5.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} (\partial_x u)^2 \,\mathrm{d}x = 2\int_0^{2\pi} -(\partial_x^2 u)^2 + \alpha(\partial_x u)^2 - 3u^2(\partial_x u)^2 \,\mathrm{d}x. \tag{5.26}$$

If $\alpha < 0$ all terms on the right-hand side are negative and we have

$$\limsup_{t\to\infty} \|u(t)\|_{H^1} = 0.$$

In order to obtain estimates which are also good for small $\alpha \geq 0$ we split the parameter regime $\alpha \geq 0$ in two parts. First let $\alpha \in [0, 1/2]$. Adding (5.25) and (5.26) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} u^2+(\partial_x u)^2\,\mathrm{d}x \\
&= 2\int_0^{2\pi} -(\partial_x^2 u)^2+(\alpha-1)(\partial_x u)^2-3u^2(\partial_x u)^2+\alpha u^2-u^4\,\mathrm{d}x \\
&\le 2\int_0^{2\pi} -(\partial_x u)^2/2+\alpha u^2-u^4\,\mathrm{d}x \\
&\le 2\int_0^{2\pi} -(\partial_x u)^2/2-u^2/2+1/8+\alpha^2/2\,\mathrm{d}x \\
&\le -\int_0^{2\pi} u^2+(\partial_x u)^2\,\mathrm{d}x+\pi.
\end{aligned}$$

This immediately shows that

$$\limsup_{t\to\infty}\|u(t)\|_{H^1}^2 \le \pi.$$

For $\alpha > 1/2$ we consider

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} 2\alpha u^2+(\partial_x u)^2\,\mathrm{d}x \\
&= 2\int_0^{2\pi} -(\partial_x^2 u)^2-\alpha(\partial_x u)^2-3u^2(\partial_x u)^2+2\alpha^2u^2-2\alpha u^4\,\mathrm{d}x \\
&\le 2\int_0^{2\pi} -\alpha(\partial_x u)^2+2\alpha^2u^2-2\alpha u^4\,\mathrm{d}x \\
&\le 2\int_0^{2\pi} -\alpha(\partial_x u)^2-2\alpha^2u^2+2\alpha^3\,\mathrm{d}x.
\end{aligned}$$

Hence, for $E=\int_0^{2\pi} 2\alpha u^2+(\partial_x u)^2\,\mathrm{d}x$ we have $E' \le -2\alpha E+8\pi\alpha^3$ and thus

$$E(t) \le \mathrm{e}^{-2\alpha t}E(0)+\frac{8\pi\alpha^3}{2\alpha}(1-\mathrm{e}^{-\alpha t}).$$

Hence, $\limsup_{t\to\infty} E(t) \le 4\pi\alpha^2$. Since $\|u\|_{H^1} \le (\int 2\alpha u^2+(\partial_x u)^2\,\mathrm{d}x)^{1/2}$ for $\alpha > 1/2$, we are done. □

5.3.2. Existence of the attractor. The existence proof of attractors in finite dimensions uses the argument that a bounded sequence contains a convergent subsequence. In infinite dimensions this is in general no longer true. Hence, this compactness argument has to be recovered by using the smoothing properties of the solution operator $\mathcal{S}_t$ with $\mathcal{S}_t u_0 = u(\cdot,t)$.

Lemma 5.3.3. *For $t > 0$ fixed the solution operator $\mathcal{S}_t$ maps bounded balls of H_0^1 into bounded balls of $H^2 \cap H_0^1$.*

Proof. We consider the variation of constant formula and estimate

$$\begin{aligned}\|u(t)\|_{H^2} \leq& \|T(t)u_0\|_{H^2} + \int_0^t \|T(t-\tau)u^3(\tau)\|_{H^2}\,\mathrm{d}\tau \\ \leq& Ct^{-1/2}\|u_0\|_{H^1} + \int_0^t C(t-\tau)^{-1/2}\,\mathrm{d}\tau \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^1}^3 < \infty, \end{aligned} \tag{5.27}$$

where $T(t)$ is the semigroup from Example 5.2.19. □

Since $H^2 \cap H_0^1$ is compactly embedded in H_0^1 we have compactness of the operator $\mathcal{S}_t$ in H_0^1. Theorem 5.3.2 thus shows that the Chafee-Infante problem (5.24) defines a dissipative dynamical system such that Theorem 2.4.4 applies.

Theorem 5.3.4. *For the Chafee-Infante problem* (5.24) *there exists a non-empty, compact, time-invariant set $\mathcal{A} = \omega(B) \subset H_0^1$, the global attractor, for which*

$$\operatorname{dist}(u(t,B),\mathcal{A}) = \sup_{b\in B}\inf_{a\in\mathcal{A}} \|u(t,b)-a\|_{H^1} \to 0 \quad \textit{for} \quad t\to\infty,$$

where B is the absorbing set from Theorem 5.3.2.

Proof. For convenience we repeat the main steps from the proof of Theorem 2.4.4. The attractor is defined by

$$\mathcal{A} = \bigcap_{t\geq 0} A_t$$

with $A_t = \operatorname{clos}_{H^1}(\mathcal{S}_t(B))$. Since B is positively invariant, the family $(A_t)_{t\geq 0}$, satisfies $A_{t_1} \subset A_{t_2}$ for $t_1 > t_2$. Hence, $\mathcal{A} \subset A_0$ is bounded. Since $\mathcal{S}_t$ is a compact operator for $t > 0$, the set A_t is compact for $t > 0$. Since $(A_t)_{t\geq 0}$ is a decreasing family of compact non-empty sets, the attractor $\mathcal{A} = \cap_{t\geq 0} A_t$ is non-empty and compact.

We skip the proof of the time invariance and restrict ourselves to the attractivity which is proved by contradiction. We assume that B is not attracted by $\mathcal{A}$. Then there exists a $\delta > 0$, sequences $t_n \to \infty$ and $u_n \in B$, such that $\operatorname{dist}(\mathcal{S}_{t_n}(u_n),\mathcal{A}) > \delta > 0$ for all $n \in \mathbb{N}$. For a small $t > 0$ the sequence $\mathcal{S}_{t_n - t}(u_n)$, $(n \in \mathbb{N})$ is bounded. Since $\mathcal{S}_t$ is a compact operator there exists a subsequence such that $v_j = \mathcal{S}_{t_{n_j}}(u_{n_j})$ converges towards a w for $j \to \infty$. Therefore, $w \in \mathcal{A}$ which contradicts the above assumption that the sequence is bounded away from $\mathcal{A}$. □

5.3.3. The choice of regularity does not matter (much). As already said, major differences between finite-dimensional systems and infinite-dimensional systems are due to the fact that in finite dimensions all norms are equivalent, whereas in infinite dimensions there are infinitely many non-equivalent norms and so infinitely many possible non-equivalent phase spaces. More or less all definitions in the theory of dynamical systems, such as continuity of solutions w.r.t. time, stability of solutions, etc. depend on the chosen norm. Therefore, we expect that the choice of a suitable phase space in infinitely many dimensions in general plays a crucial role. It is the purpose of this section to explain that for systems with smoothing properties this often is not the case. If there is a global bound in one H^θ-space, then it does not matter which H^θ-space is chosen as long as these spaces are connected with a smoothing estimate.

The estimate (5.27) can be generalized to

$$\begin{aligned}\|u(t)\|_{H^{\theta+1}} \leq & \|T(t-\tau)u(\tau)\|_{H^{\theta+1}} + \int_\tau^t \|T(t-s)u^3(s)\|_{H^{\theta+1}}\,\mathrm{d}s \\ \leq & C(t-\tau)^{-1/2}\|u(\tau)\|_{H^\theta} + \int_\tau^t C(t-s)^{-1/2}\,\mathrm{d}s \sup_{s\in[\tau,t]}\|u(s)\|_{H^\theta}^3 \\ \leq & C(t-\tau)^{-1/2}C_\theta(\tau) + 2C(t-\tau)^{1/2}C_\theta(\tau)^3,\end{aligned}$$

where $C_\theta(\tau) = \sup_{s\in[\tau,\infty]}\|u(s)\|_{H^\theta}$. Hence

$$C_{\theta+1}(t) \leq C(t-\tau)^{-1/2}C_\theta(\tau) + 2C(t-\tau)^{1/2}C_\theta(\tau)^3. \tag{5.28}$$

From Theorem 5.3.2 we know that $C_1(0) < \infty$ and that $\limsup_{\tau\to\infty} C_1(\tau) \leq R$. In H^2 we have the local existence and uniqueness of solutions, i.e., for $u_0 \in H^2$ with $\|u_0\|_{H^2} \leq \widetilde{C}$ there is a $T_2 > 0$ and a $\widetilde{C}_2 < \infty$ such that the solutions exist for all $t \in [0, T_2]$ and $\sup_{t\in[0,T_2]}\|u(t)\|_{H^2} \leq \widetilde{C}_2$. Moreover, we get

$$C_2(T_2+\tau) \leq CT_2^{-1/2}C_1(\tau) + 2CT_2^{1/2}C_1(\tau)^3 < \infty.$$

Combining the last two estimates shows that

$$\sup_{t\in[0,\infty)}\|u(t)\|_{H^2} \leq \max(\widetilde{C}_2, C_2(T_2)) < \infty.$$

Moreover, choosing $t-\tau = 1$ in (5.28) yields

$$\limsup_{t\to\infty}\|u(t)\|_{H^2} \leq CR + 2CR^3.$$

Hence, we have established an absorbing set in H^2. With the same procedure we show the existence of an absorbing set in H^3, etc.. As a consequence the attractor exists in each H^θ and the choice of phase space is not crucial with this respect.

5.3.4. Characterization and bifurcation of the attractor. Here we give a characterization of the attractor for different values of α. In a first step we write (5.24) as a gradient system, cf. §2.4.5. We have

$$\partial_t u = \partial_x^2 u + \alpha u - u^3 = -\beta\partial_u V(u)$$

with potential

$$V(u) = \int_0^\pi \frac{1}{2}(\partial_x u(x))^2 - \frac{\alpha}{2}u(x)^2 + \frac{1}{4}u(x)^4 \,\mathrm{d}x$$

and β a linear map defined below. In order to justify this formula, first recall that for a function $V : \mathbb{R}^d \to \mathbb{R}$ we have

$$V(u + \varepsilon v) = V(u) + \varepsilon a^T v + \mathcal{O}(\varepsilon^2) = V(u) + \varepsilon \langle a, v\rangle + \mathcal{O}(\varepsilon^2) \text{ for all } v \in \mathbb{R}^d,$$

where $\langle u, v\rangle = u^T v$ is the scalar product between the vectors u and v, i.e., the derivative is defined as an element of the dual space of $\mathbb{R}^d$. However, it can be identified with $\mathbb{R}^d$ through the map

$$\beta : \mathrm{Lin}(\mathbb{R}^{\mathrm{d}}, \mathbb{R}) \to \mathbb{R}^d, \qquad \langle a, \cdot\rangle \mapsto a.$$

For a map $V : X \to \mathbb{R}$ where the function space X is equipped with the scalar product

$$\langle u, v\rangle = \int_0^\pi u(x)\overline{v(x)} \,\mathrm{d}x$$

we define the map

$$\beta : \mathrm{Lin}(\mathrm{X}, \mathbb{R}) \to X, \qquad \langle a, \cdot\rangle \mapsto a.$$

This is well defined since in Hilbert spaces the dual space $\mathrm{Lin}(\mathrm{X}, \mathbb{R})$ can be identified with X by the Riesz representation theorem [**Alt16**, Satz 4.1]. Using the boundary conditions and integration by parts we find

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{V(u+\varepsilon v) - V(u)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_0^\pi (\frac{1}{2}(\partial_x(u+\varepsilon v))^2 - \frac{\alpha}{2}(u+\varepsilon v)^2 + \frac{1}{4}(u+\varepsilon v)^4) \\
&\qquad\qquad - (\frac{1}{2}(\partial_x u)^2 - \frac{\alpha}{2}u^2 + \frac{1}{4}u^4) \,\mathrm{d}x \\
&= - \int_0^\pi (\partial_x^2 u + \alpha u - u^3) v \,\mathrm{d}x
\end{aligned}$$

and so by comparison

$$\beta \partial_u V(u) = -(\partial_x^2 u + \alpha u - u^3).$$

Therefore, (5.24) is a gradient system in H_0^1, and hence the function $t \mapsto V(u(t))$ decreases along solutions $u = u(t)$, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} V(u(t)) \leq 0$, where equality only holds in fixed points. Consequently, no non-trivial periodic solution can occur. Moreover, V is bounded from below, since

$$V(u) = \int_0^\pi \frac{1}{2}(\partial_x u(x))^2 - \frac{\alpha}{2}u(x)^2 + \frac{1}{4}u(x)^4 \,\mathrm{d}x \geq - \int_0^\pi \frac{\alpha^2}{4} \,\mathrm{d}x = -\frac{\pi\alpha^2}{4}.$$

Similar to the finite-dimensional situation, cf. Theorem 2.4.15, the attractor consists of the fixed points and their unstable manifolds, cf. [**Rob01**, Theorem 10.13]. In case that only finitely many fixed points exist, the attractor consists of these fixed points and their heteroclinic connections. This can be seen directly. In a gradient system every solution must end in a fixed point. Solutions in the attractor must also start in one of the finitely many fixed points. This follows from the fact that backwards in time the system in the attractor is a gradient system, too. The potential is given by $-V$ and it is bounded on the attractor.

We compute the fixed points, or stationary solutions, of the PDE, which satisfy

$$\partial_x^2 u + \alpha u - u^3 = 0.$$

Due to the boundary conditions $u(0) = u(\pi) = 0$ in the (u, u')-plane we have to find solutions which start from the $v = u'$-axis, end on this axis, and need for this part of the orbit the 'time' $x = \pi$. For all $\alpha > 0$ the phase portrait looks qualitatively the same. The periodic orbits around the origin have a periodicity which is i) minimal at the origin, namely the periodicity of the linearization, $2\pi/\sqrt{\alpha}$, ii) infinity at the heteroclinic orbits, and iii) which increases strictly monotonic with the distance from the origin. Thus, non-trivial equilibria of (5.24) can only exist for $\alpha > 1$ since for $\alpha \leq 1$ the solutions are too slow to make half of the periodic orbit in a time π.

Using i)-iii) the complete bifurcation picture can be established in a rigorous way. The number of solutions with $u(0) = u(\pi) = 0$ changes for $m\pi/\sqrt{\alpha} = \pi$ with $m \in \mathbb{N}$, an integer multiple of half the minimal period. As a consequence, for $\alpha \in (-\infty, 1]$ we have one equilibrium, the origin; for $\alpha \in (1, 4]$ we have 3 equilibria, the origin, and two equilibria called $u_{\pm 1}$; for $\alpha \in (4, 9]$ we have 5 equilibria, $\ldots$; and for $\alpha \in (m^2, (m+1)^2]$ we have $2m + 1$ equilibria, the origin, $u_{\pm 1}, \ldots,$ and $u_{\pm m}$.

Hence, for fixed α there are only finitely many fixed points which are elements of the attractor. In order to understand the dynamics in the attractor, i.e., to find the heteroclinic connections between the fixed points, we analyze the linearization at the fixed point $u \equiv 0$, i.e.,

$$\partial_t u = \partial_x^2 u + \alpha u,$$

with $u(0, t) = u(\pi, t) = 0$, or equivalently, with $u(x, t) = \sum_{n \in \mathbb{N}} \widehat{u}_n(t) \sin(nx)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \widehat{u}_n = (\alpha - n^2) \widehat{u}_n.$$

Therefore, the linear operator

$$\Lambda \cdot = \partial_x^2 \cdot + \alpha \cdot$$

with Dirichlet boundary conditions has eigenvectors $u(x) = \sin mx$ with associated eigenvalues $\lambda = \alpha - m^2$ for $m \in \mathbb{N}$. Equivalently the infinite-dimensional diagonal matrix

$$(\widehat{\Lambda}_{nm})_{n,m \in \mathbb{N}} = ((\alpha - m^2) \delta_{nm})_{n,m \in \mathbb{N}}$$

has the eigenvectors e_m defined by $(e_m)_n = \delta_{mn}$ with associated eigenvalues $\lambda = \alpha - m^2$.

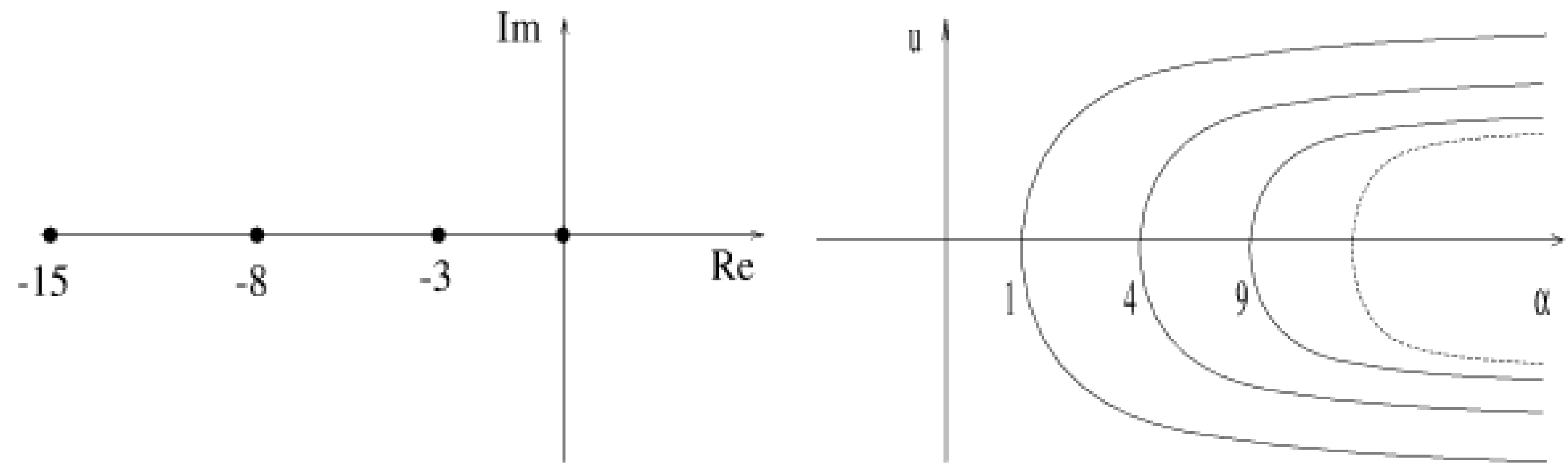


Figure 5.5. Left: the spectrum of the operator $\Lambda\cdot = \partial_x^2 \cdot + \alpha\cdot$ under Dirichlet boundary conditions for $\alpha = 1$. Right: the bifurcation diagram. At the parameter values $\alpha = n^2$ unstable equilibria bifurcate via a pitchfork bifurcation from the trivial branch $u(\alpha) \equiv 0$.

Hence, for $\alpha \in (-\infty, 1)$ the origin is asymptotically stable. For $\alpha > 1$ the origin is unstable, with a one-dimensional unstable manifold for $\alpha \in (1, 4]$, with a two-dimensional unstable manifold for $\alpha \in (4, 9]$, and with an m-dimensional unstable manifold for $\alpha \in (m^2, (m+1)^2]$.

For $\alpha \in (1, 4)$ the one-dimensional unstable manifold of the origin ends in the stable equilibria $u_{\pm 1}$. For $\alpha \in (4, 9)$ the equilibria $u_{\pm 1}$, $u_{\pm 2}$ lie on the two-dimensional unstable manifold of the origin. Since $u_{\pm 2}$ bifurcates from the unstable origin, these fixed points are also unstable and their one-dimensional unstable manifold ends in $u_{\pm 1}$.

The reasons are as follows. Since for fixed α the fixed points u_j are isolated and since the linearization only has real eigenvalues due to the gradient structure, no eigenvalue of the linearization around the equilibria u_j crosses the imaginary axis after the bifurcation when α is increased. Therefore, the dimension of the unstable manifold of u_j is the same as at their bifurcation point from the trivial branch. The fixed points $u_{\pm 1}$ bifurcating at $\alpha = 1$ are always stable. The fixed points $u_{\pm 2}$ bifurcating at $\alpha = 4$ have a one-dimensional unstable manifold which ends in the fixed points $u_{\pm 1}$. The fixed points $u_{\pm 3}$ bifurcating at $\alpha = 9$ have a two-dimensional unstable manifold and so heteroclinic connections to the fixed points to $u_{\pm 1}$ and $u_{\pm 2}$ exist. Figure 5.6 sketches the dynamics in the attractor of (5.24). For $\alpha \in (n^2, (n+1)^2)$ we have an attractor of dimension n consisting of finitely many fixed points and heteroclinic orbits between these fixed points, in particular, it contains the n-dimensional unstable manifold of the origin. A local bifurcation analysis via center manifold reduction can be found in §13.2.1.

Further Reading. Our point of view of PDEs over bounded sets as countably many ODEs is similar to [**Hal88, Rob01, KP13**], while [**Paz83**] gives

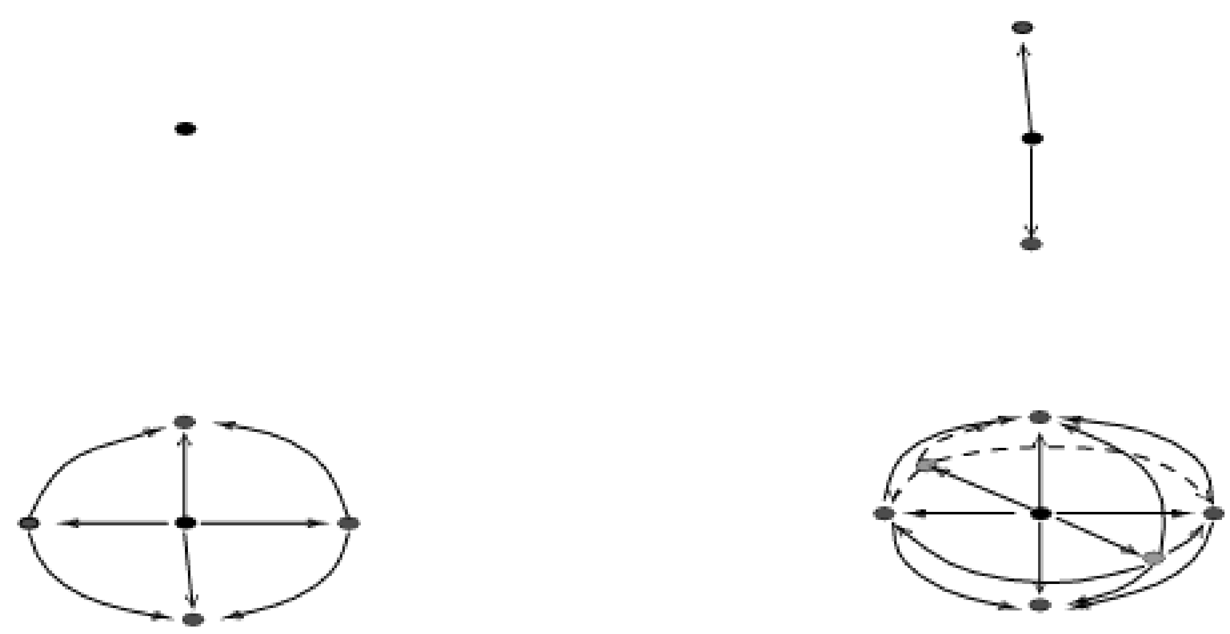


Figure 5.6. The finite-dimensional attractors consisting of fixed points and connecting orbits for $\alpha < 1$, $1 < \alpha < 4$, $4 < \alpha < 9$, and $9 < \alpha < 16$. Only selected heteroclinic orbits have been plotted. The attractor always contains the n-dimensional unstable manifold of the origin.

an excellent and concise account of the general semigroup approach, see also [**RR04**, Chapter 11]. Fourier series can be found in most textbooks on analysis and functional analysis, and in many books on PDE; we recommend [**Olv14**, Chapter 3] for an introduction with a PDE point of view, and [**Duo01**] for a concise but comprehensive treatment. Classical books on linear functional analysis, covering much wider ground than what is used here and in the following chapters are [**Yos71, RS75a, Kat95**]; our favorites are [**Rud73, Wer00, Alt16**]. A concise introduction to nonlinear functional analysis is [**AA11**]. Our presentation of function spaces follows [**Alt16, Wlo87**], but the same material can be found in many textbooks, for instance, from a PDE perspective, in [**Str92, RR04, Eva98, Sal08**]. Comprehensive treatments of distributions and Sobolev spaces, including various versions of Poincaré inequalities and Sobolev imbeding theorems, are given in [**Hör83, Maz11**], and [**Tay96**, Chapters 3,4,13]. See also [**Geo15**] for a concise introduction aimed at graduate students, and Section 7.3 of this book for the case of unbounded domains.

Exercises

5.1. Prove that the space $c_{00} = \{u : \mathbb{Z} \to \mathbb{R} : u_n \neq 0 \text{ for finitely many } n\}$ equipped with the ℓ_1-norm is not complete.

5.2. Consider $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} u_n = -\omega_n^2 u_n$, $n \in \mathbb{N}$, with $u_n(t) \in \mathbb{R}$, and $\omega_n \in \mathbb{R}$. Write the equation as first order system and find some phase space where the infinitely many first order ODEs define a C_0-semigroup. Under which additional assumptions on the ω_n is the semigroup uniformly continuous, differentiable, or analytic?

5.3. Work out the details for Example 5.1.15.

5.4. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and (e_n) an orthonormal basis of H. Let $\lambda_n > 0$, $\lambda_n \le \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = \infty$. Finally, let

$$V := \{u \in H : \sum_{n\in\mathbb{N}} \lambda_n \langle u, e_n\rangle^2 < \infty\}.$$

Show that $a(u,v) = \sum_{n\in\mathbb{N}} \lambda_n \langle u, e_n\rangle \langle v, e_n\rangle$ defines a scalar product in V such that V is a Hilbert space, and $V \subset H$ compact.

5.5. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Let $\lambda_n > 0$, $\lambda_n \le \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = \infty$. For $t \ge 0$ let $T(t) : H \to H$ be defined by $T(t)u = \sum_{n\in\mathbb{N}} \mathrm{e}^{-\lambda_n t} \langle u, e_n\rangle e_n$. Show that $(T(t))$ is a C_0-semigroup, and that $\lim_{t\searrow 0} \frac{1}{t}(T(t)u - u) =: Au$ exists iff $\sum_{n\in\mathbb{N}} \lambda_n^2 \langle u, e_n\rangle^2 < \infty$.
Finally, for $v = \sum_{n\in\mathbb{N}} \lambda_n \langle u, e_n\rangle e_n$ show that $T(t)u - u = \int_0^t T(s)v\,\mathrm{d}s$.

5.6. Prove local existence and uniqueness of solutions for $\partial_t u_n = -n^4 u_n + u_n^3$, with $n \in \mathbb{Z}$, in spaces $\ell_{p,\theta}$. Do we have global existence and uniqueness of solutions?

5.7. Consider the discrete NLS equation

$$\mathrm{i}\partial_t u_n = \varepsilon(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^2 u_n,$$

with $u_n(t) \in \mathbb{C}$ and $n \in \mathbb{Z}$ for $0 \le \varepsilon \le 1$.

a) Show the local existence and uniqueness of solutions in ℓ_2. Note that the linear part is not diagonal, but bounded. Show that the system conserves the ℓ_2-norm. Conclude the global existence and uniqueness of solutions in ℓ_2 from this fact.

b) For $\varepsilon = 0$ find non-trivial solutions U_0 of the form $u_0(t) = r_0 \mathrm{e}^{\mathrm{i}\omega t}$ with $r_0 \in \mathbb{R}$ and $u_n = 0$ for all other $n \in \mathbb{Z}$. Use the implicit function theorem for instance in ℓ_∞ to prove that for $\varepsilon > 0$ there are solutions U_ε of the form $u_n(t) = r_n \mathrm{e}^{\mathrm{i}\omega t}$ nearby U_0. What additional information is gained if $\ell_{\infty,\theta}$ instead of ℓ_∞ is used?

5.8. Show that any $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is locally Lipschitz continuous.

5.9. a) Let $I \subset \mathbb{R}$ be a compact interval and $\alpha \in (0,1)$. Show that $C^{0,\alpha}(I)$ is a Banach space.
b) Let $I = [0,1]$ and $0 < \beta < \alpha < 1$. Show that $C^{0,\alpha}(I) \subset C^{0,\beta}(I)$ as a proper subset.

5.10. Let $\Omega = (0,1)$. Find sequences (u_n), $u_n : \Omega \to \mathbb{R}$ such that:
(a) (u_n) bounded in $H^1(\Omega)$, but (u_n) does not converge in $L^2(\Omega)$.
(b) $u_n \to 0$ in $L^2(\Omega)$ but $u_n(x) \not\to 0$ for all $x \in \Omega$.
(c) $\|u_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$ and $u_n \to 0$ weakly in $L^2(\Omega)$.
Remark. Since $H^1(\Omega) \subset L^2(\Omega)$ compact, in (a) we always have $u_{n_k} \to v$ in $L^2(\Omega)$ for a subsequence u_{n_k}.

5.11. Let $\Omega = B_{1/2}^{\mathbb{R}^2}(0)$. Show that $u(x) = \ln|\ln|x|| \in H^1(\Omega) \setminus L^\infty(\Omega)$.

5.12. Let $d \in \mathbb{N}$ and $\Omega = B_1(0)$ in $\mathbb{R}^d$. For which α do we have (a) $|x|^\alpha \in H^1(\Omega)$; (b) $(\sin|x|)^\alpha \in H^1(\Omega)$; (c) $(\ln|x|)^\alpha \in H^1(\Omega)$?

5.13. For $H = L^2(0,1)$ define $F : H \to \mathbb{R}$ by $F(u) = \int_0^{1/2} u(x)\,\mathrm{d}x$. Do we have $F \in H'$? If so, find a representation $\langle\cdot, v\rangle_{L^2}$ of F with $v \in H$.

5.14. For the following PDEs with $x \in (0, 2\pi)$ and periodic boundary conditions investigate whether the solution operator defines a C_0-semigroup in $L^2_{per}((0, 2\pi), \mathbb{R})$ with smoothing properties

$$a)\ \partial_t u = \partial_x^4 u, \qquad b)\ \partial_t u = -\partial_x^4 u, \qquad c)\ \partial_t u = \partial_x^3 u.$$

5.15. Consider the complex GL equation

$$\partial_t u = (1 + \mathrm{i}\alpha)\partial_x^2 u + Ru - (1 + \mathrm{i}\beta)\,|u|^2\,u$$

with 2π-periodic boundary conditions, $u(x,t) \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{R}$. In case $|\beta| < 1/3$ prove the global existence of solutions in H^1 for all $R \in \mathbb{R}$.

5.16. Consider $\partial_t u = \partial_x^2 u + u^3$ for $t \geq 0$, $x \in (0, \pi)$ and $u(x,t) \in \mathbb{R}$ with boundary condition $u(0,t) = u(\pi,t) = 0$ and initial condition $u(x,0) = \phi(x)$, cf. [**Hen81**, Page 49]. Prove that there are solutions which converge in finite time towards ∞. Hint: Derive a differential inequality for $s(t) = \int_0^\pi \sin(x)u(x,t)dx$. With Hölder's inequality we obtain $s(t) \leq 2^{2/3}(\int_0^\pi \sin(x)u^3(x,t)dx)^{1/3}$ and so $\frac{\mathrm{d}}{\mathrm{d}t}s \geq -s + s^3/4$.

5.17. Write $\partial_t u = -\partial_x^4 u + \sin(u)$, with $u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, and $t \geq 0$, subject to periodic boundary conditions $u(x,t) = u(x + 2\pi, t)$ as a gradient system.

5.18. Consider the Cahn-Hilliard equation $\partial_t u = \partial_x^2(-\gamma\partial_x^2 u - u + u^3)$, with $u(x,t) \in \mathbb{R}$, $\gamma > 0$, and 2π-periodic boundary conditions.

a) Prove that $\frac{\mathrm{d}}{\mathrm{d}t}C = 0$, where $C(t) = \int_0^{2\pi} u(x,t)\,\mathrm{d}x$.

b) Let $F(u) = \int_0^{2\pi} \frac{1}{4}(u^2 - 1)^2 + \frac{\gamma}{2}\,|\partial_x u|^2\ \mathrm{d}x$ and show that $\frac{\mathrm{d}}{\mathrm{d}t}F = -\int |\nabla w|^2\ \mathrm{d}x$ with $w = u^3 - u - \gamma\partial_x^2 u$.

c) Find the possible ω-limit sets.

The Navier-Stokes equations

6.1. Introduction

In this chapter we give an introduction to the Euler and Navier-Stokes equations, which over unbounded domains will also play a role in subsequent chapters. The global existence and uniqueness of solutions of the three-dimensional (3D) Navier-Stokes equations is one of the seven so called 'one million dollar' or millennium problems in mathematics presented by the Clay Mathematics Institute in the year 2000. There are a number of reasons for this choice. On the one hand, the solution of this problem would allow us to understand and simulate the motion of fluids more rigorously. On the other hand, the 3D Navier-Stokes equations are interesting PDEs which resisted so far all attempts to prove the global existence and uniqueness of solutions.

Their history goes back a long way. The equations describing the motion of non-viscous fluids are called Euler equations and have been derived by Leonhard Euler (1707–1783). The Navier-Stokes equations generalize the Euler equations and include the case of viscous fluids. They have been derived independently by a number of people, including Claude-Louis Navier (1785–1839), George Stokes (1819–1903), Simeon Poisson (1781–1840) and Jean Claude Saint-Venant (1797–1886).

First we recall the derivation of the Navier-Stokes equations, following [**Fow97**, §6]. Then we focus on the analysis of the Navier-Stokes equations in $\Omega = \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, i.e., $\Omega = [0, 2\pi)^d$ with periodic boundary conditions. After the presentation of some local existence and uniqueness results we formulate the global existence question. The local existence and uniqueness of solutions in some phase space X is obtained by a fixed point argument in $C([0,T_0],X)$ for a $T_0>0$, using the variation of constant formula. A good choice of the phase space X turns out to be essential in this construction. The background of the millennium problem is the fact that in infinite-dimensional spaces there are infinitely many non-equivalent norms. For the 3D Navier-Stokes equations so far in no phase space simultaneously the global existence of solutions and their uniqueness can be shown.

6.1.1. Derivation of the Navier-Stokes equations. The Navier-Stokes equations describe the velocity and the pressure field of an incompressible fluid. By Newton's law (force=mass×acceleration) the N molecules of the fluid satisfy the system of ODEs

$$m\ddot{x}_j = F_j(x_1,\dots,x_N)$$

for $j=1,\dots,N$. The motion of the fluid is completely determined by the evolution of this system. However, the system is pretty useless due to the very large number N. Therefore, the fluid is modeled as a continuum. In doing so we have to guarantee that no molecules are lost, i.e., that mass is conserved.

The velocity field of the continuum at a position $x\in\mathbb{R}^d$ at a time t is denoted by $u(x,t)\in\mathbb{R}^d$ for $d=2,3$. With $\rho=\rho(x,t)\in\mathbb{R}$ we denote similarly the density of the fluid. The Navier-Stokes equations consist of two equations, a scalar one for the conservation of mass and a second equation with d components for the conservation of the momentum. In general, by the internal friction of the fluid heat will be produced which leads to a coupling of the Navier-Stokes equations with a heat equation. However, here we will neglect this aspect.

Conservation of mass. We consider a fixed test volume V with surface S. The total mass in V can only change by the flow through the boundary S, i.e.,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V &= \int_V \partial_t\rho\,\mathrm{d}V = -\int_S \rho u\cdot n\,\mathrm{d}S = -\int_S \sum_{j=1}^d \rho u_j n_j\,\mathrm{d}S \\ &= -\int_V \mathrm{div}(\rho u)\,\mathrm{d}V = -\int_V \sum_{j=1}^d \partial_{x_j}(\rho u_j)\,\mathrm{d}V,\end{aligned}$$

where we used the Gauss integral theorem and where $n(x)=(n_1,\dots,n_d)(x)$ is the outer unit normal in the point x at the boundary S. Since this relation holds for all test volumes V the integrands must be equal, i.e.,

$$\partial_t\rho + \mathrm{div}(\rho u) = 0. \tag{6.1}$$

Conservation of momentum. Similarly, the momentum of a test volume V can only change by the flow through the boundary and by forces, for instance friction forces, on the surface of the test volume. For these forces f we assume the existence of a matrix $\sigma = (\sigma_{ij})_{i,j=1,\ldots,d}$, the so called stress tensor, which relates the direction of the outer normal n with the direction and magnitude of the resulting force f, i.e.,

$$f_i = \sum_{j=1,\ldots,d} \sigma_{ij} n_j.$$

For examples see below. Hence, we obtain for the change of the momentum

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho u_i \,\mathrm{d}V = -\int_S \sum_{j=1}^{d} (\rho u_i) u_j n_j \,\mathrm{d}S + \int_S \sum_{j=1}^{d} \sigma_{ij} n_j \,\mathrm{d}S.$$

Application of the Gauss integral theorem and the above arguments yield

$$\partial_t(\rho u_i) + \operatorname{div}(\rho u_i u) = \operatorname{div}(\sigma_{i\cdot}).$$

Using conservation of mass gives

$$\partial_t(\rho u_i) = \rho \partial_t u_i + u_i \partial_t \rho = \rho \partial_t u_i - u_i \sum_{j=1,\ldots,d} \partial_{x_j}(\rho u_j),$$

thus, in vector notation,

$$\rho[\partial_t u_i + (u \cdot \nabla) u_i] = \nabla \cdot \sigma_{i\cdot} \tag{6.2}$$

or, in coordinates,

$$\rho \left[\partial_t u_i + \sum_{j=1}^{d} (u_j \cdot \partial_{x_j}) u_i \right] = \sum_{j=1}^{d} \partial_{x_j} \sigma_{ij}.$$

Constitutive laws. In order to obtain a closed set of equations from (6.1) und (6.2) we need to know how the stress tensor σ depends on the velocity u and the density ρ. Such a relation $\sigma = \sigma(u, \rho)$ is called a constitutive law and depends on the fluid under consideration, i.e., the function differs strongly for instance between water and honey. It is possible that $\sigma(t)$ is not only a function of $(u, \rho)(t)$, but depends on the whole history of (u, ρ), cf. [**Ren00**].

i) For a non-viscous fluid, i.e., for a fluid without internal friction, the surface forces f are only due to pressure. Hence, the forces f are parallel to the outer normals n and their magnitude does not depend on the direction, i.e.,

$$f_i = \sum_{j=1}^{d} \sigma_{ij} n_j = -\sum_{j=1}^{d} p \delta_{ij} n_j = -p n_i, \tag{6.3}$$

which yields the constitutive law

$$\sigma_{ij} = -p\delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ else. Since

$$\sum_{j=1}^{d} \partial_{x_j} \sigma_{ij} = -\sum_{j=1}^{d} \partial_{x_j}(p\delta_{ij}) = -\partial_{x_i} p$$

we obtain for non-viscous fluids

$$\partial_t u + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p. \tag{6.4}$$

The system consisting of (6.1) and (6.4) is still not closed. We have to relate p with the velocity u and density ρ. A typical choice would be $p = c\rho^\gamma$ with some constants $c > 0$ and $\gamma \geq 1$. In case of incompressible fluids, i.e., $\rho = const.$, we obtain Euler's equations

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u &= -\frac{1}{\rho}\nabla p, \\ \nabla \cdot u &= 0. \end{aligned} \tag{6.5}$$

ii) For a viscous fluid the constitutive law is given by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij},$$

where τ_{ij} will model the internal friction. In order to find a model for τ_{ij} as simple as possible we consider a stationary constant density shear flow with velocity field $(u_1(x_2), 0)$, see Figure 6.1. The friction forces act on the top surface and bottom surface, and they are proportional to the difference of these velocities. Hence, for an infinitesimal small test volume we find that the surface force f is proportional to the so called strain $\partial_{x_2} u_1$. Since the friction forces are perpendicular to the top surface we find

$$\tau_{12} = \mu \partial_{x_2} u_1,$$

where $\mu > 0$ is called the dynamic viscosity. Due to the isotropy of the fluid σ must be symmetric. This finally leads to

$$\tau_{ij} = 2\mu\dot{\varepsilon}_{ij} + \lambda \sum_{k=1}^{d} \dot{\varepsilon}_{kk}\delta_{ij}$$

where

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j).$$

The constants μ for the shear and λ for the compressions are called the Lamé-constants.

Remark 6.1.1. There is no reason why in our modeling σ should only depend linearly on the first derivatives of u. An answer to the millennium problem can lead to some corrections at this point of the modeling. ⌋

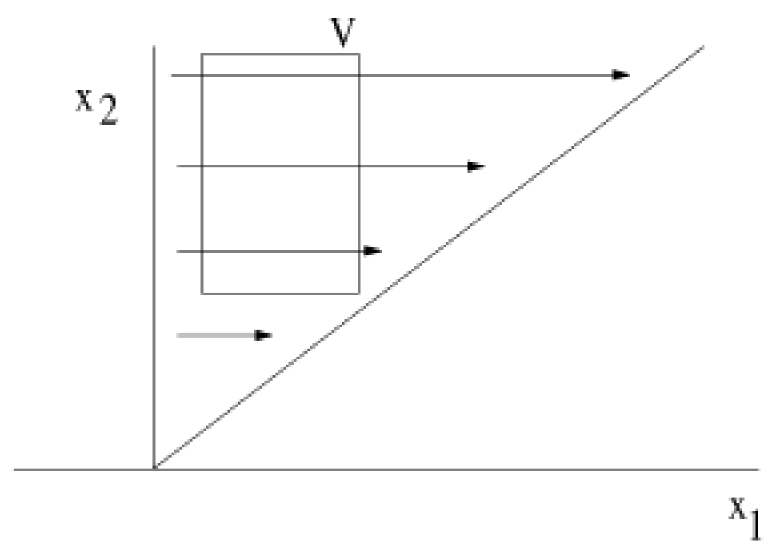


Figure 6.1. Shear flow: the internal friction is proportional to $\partial_{x_2} u_1$.

The Navier-Stokes equations. For air, compressibility is an important issue. However, for fluids such as water, ρ can be considered to be a constant, i.e., $\partial_t \rho = 0$. As in (6.5) the conservation of mass then simplifies to

$$\sum_{j=1}^{d} \partial_{x_j} u_j = \nabla \cdot u = \mathrm{div} u = 0.$$

From $\sum_{k=1}^{d} \dot{\varepsilon}_{kk} = \sum_{k=1}^{d} \partial_{x_k} u_k = 0$ we obtain $\tau_{ij} = 2\mu \dot{\varepsilon}_{ij}$. Moreover, we find

$$\sum_{j=1}^{d} \partial_{x_j} \tau_{ij} = \sum_{j=1}^{d} \partial_{x_j} (\partial_{x_j} u_i + \partial_{x_i} u_j) = \sum_{j=1}^{d} \partial_{x_j}^2 u_i + \partial_{x_i} (\sum_{j=1}^{d} \partial_{x_j} u_j) = \sum_{j=1}^{d} \partial_{x_j}^2 u_i,$$

such that

$$\partial_t u + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \Delta u,$$

where $\nu = \mu/\rho$ is called the kinematic viscosity. The Navier-Stokes equations are then given by

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -\frac{1}{\rho} \nabla p + \nu \Delta u, \\ \nabla \cdot u &= 0. \end{aligned} \tag{6.6}$$

In order to handle these equations as a dynamical system boundary conditions have to be added. At rigid boundaries, a viscous fluid satisfies $u = 0$. At free surfaces, boundary conditions involve for instance the prescription of stresses. In this section we consider the Navier-Stokes equations with periodic boundary conditions. These have no physical meaning, but allow us to focus on the equations itself.

In order to eliminate the physical units from the Navier-Stokes equations, let U be a typical velocity and l be a typical length of the flow. We set

$$u = U u^*, \quad x = l x^*, \quad p = \rho U^2 p^*, \quad t = l t^*/U,$$

and obtain after dropping the * the Navier-Stokes equations in dimensionless form

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \frac{1}{\mathcal{R}}\Delta u, \qquad \nabla \cdot u = 0,$$

where $\mathcal{R} = Ul/\nu$ is called the Reynolds number. The larger $\mathcal{R}$, the more complex the flow. Until further notice we assume w.l.o.g. for our purposes that $\mathcal{R} = 1$.

6.1.2. The vorticity, and some explicit solutions. The vorticity ω of the 3D velocity field u is defined by

$$\omega = \nabla \times u = \begin{pmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix},$$

while for 2D flows the vorticity is the scalar $\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1$. Applying the curl-operator $\nabla\times$ to the Navier-Stokes equations gives in $\mathbb{R}^3$ that

$$\partial_t \omega = \nu \Delta \omega - (u \cdot \nabla)\omega + (\omega \cdot \nabla)u. \tag{6.7}$$

Specializing (6.7) to two-dimensional flows shows $\omega \perp \nabla u$, and so in $\mathbb{R}^2$ we have

$$\partial_t \omega = \nu \Delta \omega - (u \cdot \nabla)\omega. \tag{6.8}$$

The pressure gradient has vanished from (6.7) and (6.8). On the other hand, the velocity u still appears and has to be reconstructed from ω by solving the PDEs

$$\omega = \nabla \times u \qquad \text{and} \qquad \nabla \cdot u = 0.$$

If $\Omega = \mathbb{R}^d$ the solution is given by the Biot-Savart law [**FLS64**, II-14-10] from magnetostatics, cf. Exercise 6.3.

There are major differences between the 2D case and the 3D case. Beside the diffusion and transport of vorticity which appear in 2D and 3D, in 3D there is also the production term $(\omega\cdot\nabla)u$ for vorticity. Hence, it is likely that the differences in the global existence and uniqueness question in 2D and 3D are not only an artificial functional analytic problem. In fact, experiments and simulations, cf. the discussion in [**GW06a**], show that in 3D smaller and smaller vortices are created, whereas in 2D the smaller vortices vanish and are eaten up by the larger ones.

In $\mathbb{R}^d$ or $\mathbb{T}^d$ we have that $\omega = 0$ is a solution of the vorticity equation. The fact that vorticity is preserved by the motion of the fluid together with the incompressibility of the fluid allows us to construct a number of non-trivial solutions for the Navier-Stokes equations. The existence of a potential $\Phi : \mathbb{R}^d \to \mathbb{R}$ with $u = \nabla\Phi$ follows from $\omega = \nabla \times u = 0$. The potential Φ satisfies $\Delta\Phi = 0$ due to the incompressibility of the fluid. Such flows are called potential flows. However, in general the boundaries of the physical domain will create vorticity.

Remark 6.1.2. The conditions

$$\nabla \times u = 0, \qquad \text{and} \qquad \nabla \cdot u = 0 \tag{6.9}$$

can be interpreted in two space dimensions, i.e. $u = (u_1, u_2)$, as the Cauchy-Riemann differential equations of a complex valued function $z \mapsto w(z)$ defined through $w(x_1 + \mathrm{i}x_2) = u_1(x_1, x_2) - \mathrm{i}u_2(x_1, x_2)$. From complex analysis it is well known that the complex differentiability of w, together with (6.9), implies the analyticity of w, respectively u. ⌋

Here are a number of examples for potential flows. Further interesting and more complicated exact solutions of the Navier-Stokes equations can for instance be found in [**MB02**].

Example 6.1.3. We already encountered the explicit constant shear flow solution (2D)

$$u(x,t) = \begin{pmatrix} u_1(x_2) \\ 0 \end{pmatrix} = \begin{pmatrix} cx_2 \\ 0 \end{pmatrix}, \qquad p(x,t) = p_0,$$

with p_0 some constant. This is an example of a parallel or laminar flow with non-vanishing vorticity. Further examples are Couette and Poiseuille flow. The latter describes the flow in an infinitely long pipe $\Omega = \mathbb{R} \times \Sigma$, where $\Sigma \subset \mathbb{R}^d$ is a bounded cross-section with rigid boundary conditions $u|_{\partial\Sigma \subset \mathbb{R}^d} = 0$. As an explicit example we again consider the 2D case and set $\Sigma = (-1, 1)$. Then

$$u(x,t) = \begin{pmatrix} u_1(x_2) \\ 0 \end{pmatrix}, \quad \text{with} \quad u_1(x_2) = c(x_2^2 - 1), \quad c \in \mathbb{R},$$

is an exact solution, see Figure 6.2a), with the pressure given by $p(x,t) = p_0 - \frac{c}{\mathrm{R}} x_1$. ⌋

An important property of all parallel flows is that the nonlinear term $(u \cdot \nabla)u$ of the Navier-Stokes equations drops out. As a consequence, in $\mathbb{R}^d$ parallel flows can always be superimposed. This also works in general domains if the boundary conditions permit it.

Example 6.1.4. In 2D so called irrotational strain flows are given by

$$u(x,t) = \gamma \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}, \quad p(x,t) = p_0 - \frac{\gamma^2}{2}(x_1^2 + x_2^2),$$

see Figure 6.2b), while vortices are given by

$$u(x,t) = \omega_0 \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad p = p_0 + \frac{\omega_0^2}{2}(x_1^2 + x_2^2),$$

see Figure 6.2c). A 3D flow generalizing the 2D strain flow is the irrotational stationary jet

$$u(x,t) = \begin{pmatrix} -\gamma_1 x_1 \\ -\gamma_2 x_2 \\ (\gamma_1+\gamma_2)x_3 \end{pmatrix}, \quad p(x,t) = p_0 - \frac{1}{2}(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + (\gamma_1+\gamma_2)^2 x_3^2).$$

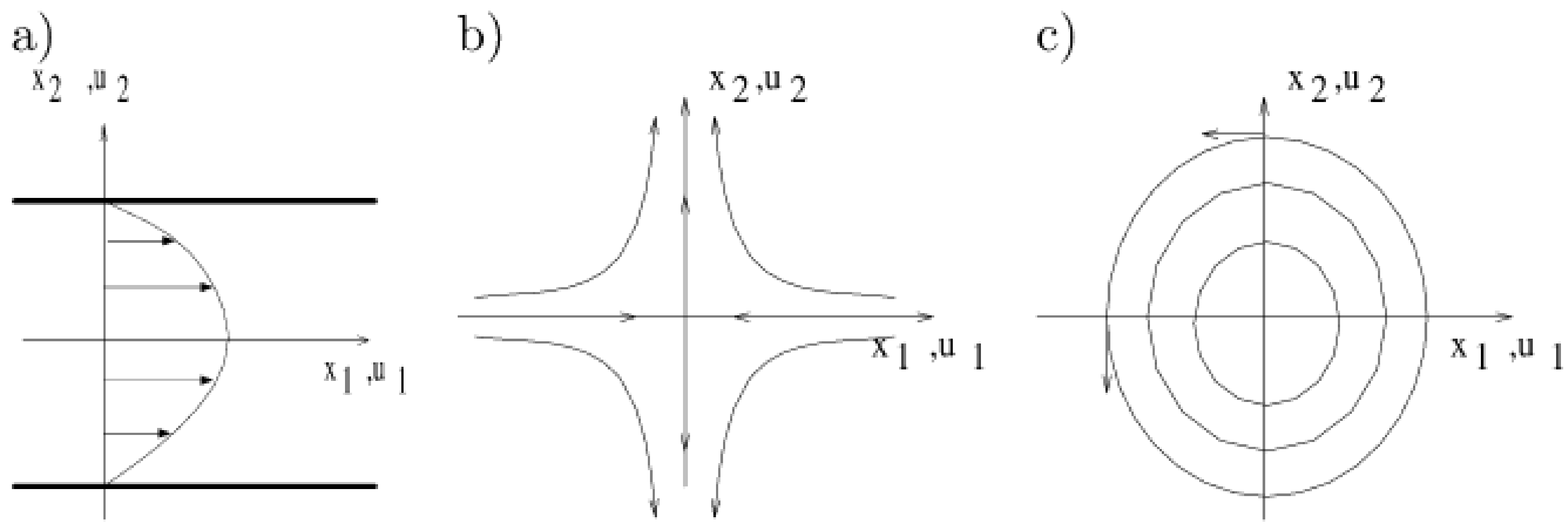


Figure 6.2. Three planar exact solutions: Poiseuille flow, strain flow, a vortex.

⌋

To get a feeling for the behavior of general solutions of the Navier-Stokes equation we recommend to do some numerical experiments. For this we refer to [**Uec09**] and the `matlab` scripts provided and explained therein, including some brief discussion of turbulence.

6.2. The equations on a torus

6.2.1. Local existence and uniqueness. We remind the reader of basic problems with existence and uniqueness of solutions of ODEs and PDEs. In Example 2.2.4 we saw that the scalar equation $\dot{u} = \sqrt{|u|}$ with the initial condition $u(0) = 0$ has infinitely many different solutions. Two examples are $u(t) = 0$ and $u(t) = t^2/4$. In Example 2.2.5 we saw that the unique solution $u(t) = \tan(t)$ of $\dot{u} = (1 + u^2)$ with the initial condition $u(0) = 0$ becomes unbounded for $t = \pi/2$.

We study the local existence and uniqueness of solutions of the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \Delta u, \qquad \nabla \cdot u = 0. \tag{6.10}$$

We follow the formulation of the millennium problem [**Fef06**] and consider the Navier-Stokes equations in $\mathbb{T}^d$, i.e., in $[0, 2\pi)^d$ with periodic boundary conditions. The phase space is chosen in such a way that its elements satisfy the boundary conditions. The Navier-Stokes equations have the special difficulty that the second equation, $\nabla \cdot u = 0$, is without a time derivative, and that the variable p occurs without time derivative at all. This problem is solved by prescribing the equation $\nabla \cdot u = 0$ as additional condition in the definition of the phase space. The term $-\nabla p$ in the first equation will be interpreted as projection P onto the divergence free vector fields such that the Navier-Stokes equations can be written as

$$\partial_t u = P\Delta u - P(u \cdot \nabla)u.$$

For periodic boundary conditions we will have $P\Delta = \Delta P$ such that we finally have to consider $\partial_t u = \Delta u - P(u \cdot \nabla)u$ in the space of divergence free vector fields $\{u : u = Pu\}$.

The Navier-Stokes equations are semi-linear parabolic differential equations such that for the construction of local solutions in time we use again the variation of constant formula

$$u(t) = \mathrm{e}^{t\Delta}u(0) - \int_0^t \mathrm{e}^{(t-s)\Delta}(P(u \cdot \nabla)u)(s)\,\mathrm{d}s \tag{6.11}$$

and the scheme introduced in §5.1.4. The semigroup $\mathrm{e}^{t\Delta}$ generated by the linear part is smoothing, i.e., $u_0 \in L^2$ implies that $t^{m/2}\partial_x^m u(t)$ is bounded in $L^2(\mathbb{T}^d)$ for every $t > 0$ and $m \in \mathbb{N}$. Semi-linear means here that the nonlinearity only contains terms with less derivatives than in the linear part, i.e., for the Navier-Stokes equations first derivatives in $(u \cdot \nabla)u$ compared with the second order derivatives in Δu. More precisely, we will prove that $\mathrm{e}^{t\Delta}$ maps H^m into H^{m+1} with a singularity $t^{-1/2}$ and that $P(u \cdot \nabla)u$ is a bilinear map from $H^{m+1} \times H^{m+1} \to H^m$. Then all assumptions following (5.21) are satisfied and the local existence and uniqueness Theorem 5.2.22 will apply. Hence, the major step is to give a precise definition of P and to investigate its analytic properties.

The projection on the divergence free vector fields. We define the projection P via the solution $v = Pf$ of the system of PDEs

$$v + \nabla p = f, \quad \nabla \cdot v = 0, \tag{6.12}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ should be thought as a placeholder for the nonlinear terms $(u \cdot \nabla)u$. For notational simplicity we carry out the analysis only in case $x \in \mathbb{R}^2$ with 2π-periodic boundary conditions. In order to solve (6.12) we make an expansion in Fourier series

$$v_j(x) = \sum_{k\in\mathbb{Z}^2} \widehat{v}_{j,k}\mathrm{e}^{\mathrm{i}k\cdot x}, \quad p(x) = \sum_{k\in\mathbb{Z}^2} \widehat{p}_k\mathrm{e}^{\mathrm{i}k\cdot x}, \quad f_j(x) = \sum_{k\in\mathbb{Z}^2} \widehat{f}_{j,k}\mathrm{e}^{\mathrm{i}k\cdot x},$$

with $\widehat{v}_{j,k}$, $\widehat{f}_{j,k}$, $\widehat{p}_k \in \mathbb{C}$ for $j = 1, 2$. Plugging this into (6.12) yields

$$\widehat{v}_{1,k} + \mathrm{i}k_1\widehat{p}_k = \widehat{f}_{1,k}, \quad \widehat{v}_{2,k} + \mathrm{i}k_2\widehat{p}_k = \widehat{f}_{2,k}, \quad \mathrm{i}k_1\widehat{v}_{1,k} + \mathrm{i}k_2\widehat{v}_{2,k} = 0. \tag{6.13}$$

In case $|k| \neq 0$ we find the solution

$$\begin{pmatrix} \widehat{v}_{1,k} \\ \widehat{v}_{2,k} \\ \widehat{p}_k \end{pmatrix} = \frac{1}{k_1^2 + k_2^2} \begin{pmatrix} k_2^2 & -k_1 k_2 & -\mathrm{i}k_1 \\ -k_1 k_2 & k_1^2 & -\mathrm{i}k_2 \\ -\mathrm{i}k_1 & -\mathrm{i}k_2 & 1 \end{pmatrix} \begin{pmatrix} \widehat{f}_{1,k} \\ \widehat{f}_{2,k} \\ 0 \end{pmatrix}.$$

For the subspace $k_1 = k_2 = 0$ there are two possibilities.

Case i). We prescribe the periodicity of the pressure p. Then we have that $\widehat{v}_{j,0} = \widehat{f}_{j,0}$ in (6.13) and that $\widehat{p}_0$ is arbitrary which is no problem since only ∇p occurs in the Navier-Stokes equations. This choice can lead to a non-vanishing mean flow.

Case ii). We require that the mean flows $\int_\Omega v_j(x_1, x_2)\,\mathrm{d}x$ vanish for $j = 1, 2$, i.e., $\widehat{v}_{1,0} = \widehat{v}_{2,0} = 0$. In order to do so we consider a pressure

$$p(x,t) = \sum_{j=1}^{2} \alpha_j x_j + \widetilde{p}(x,t),$$

where $\widetilde{p}(x,t)$ is 2π-periodic w.r.t. the x_j. Then $\partial_{x_j} p(x,t) = \alpha_j + \partial_{x_j} \widetilde{p}(x,t)$ and so $\widehat{v}_{j,0} + \alpha_j = \widehat{f}_{j,0}$. Thus, to a $\widehat{f}_{j,0}$ we always find an α_j such that $\widehat{v}_{j,0} = 0$.

Example 6.2.1. To illustrate the difference between Case i) and Case ii), we consider the vector field

$$f(x_1, x_2) = \mathrm{e}^{-2(x_1-\pi)^2 - 2(x_2-\pi)^2} \begin{pmatrix} 2 + \tanh(x_2 - \pi) \\ 0 \end{pmatrix}.$$

In Figure 6.3 we show the different effects of choosing i) or ii). ⌋

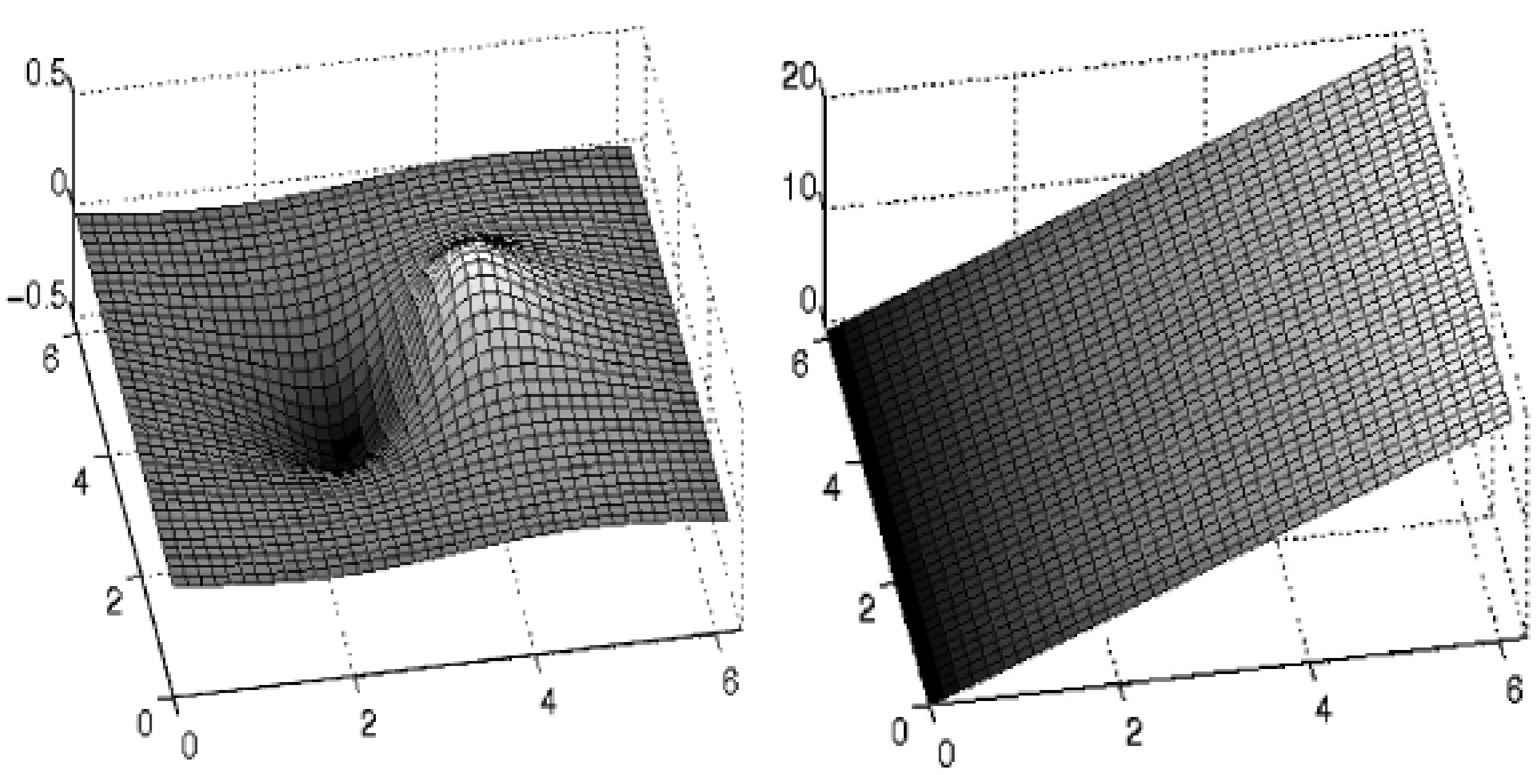


Figure 6.3. Illustration of the difference between cases i) and ii), concerning the boundary conditions for the pressure in the definition of the projection $u = Pf$ for f from Remark 6.2.1, via a plot of the pressure function. In the left panel we require a periodic pressure, giving a mean flow in Pf, while in the right panel we require zero mean flow, giving a linear growth of the pressure p.

Choosing between i) and ii) is a question of modeling. i) has the disadvantage that if a constant mean force is added to the Navier-Stokes equations then this choice leads to unbounded growth of (laminar) mean flows. Therefore, in the following we opt for ii), i.e., $\widehat{v}_{j,0} = 0$ for $j = 1, 2$ and define the projection $\widehat{P}$ as direct sum of the projections $\widehat{P}_k$, i.e., $\widehat{v}_k = (\widehat{P}\widehat{f})_k = \widehat{P}_k \widehat{f}_k$, where

$$\widehat{v}_0 = 0 \text{ and } \begin{pmatrix} \widehat{v}_{1,k} \\ \widehat{v}_{2,k} \end{pmatrix} = \frac{1}{k_1^2 + k_2^2} \begin{pmatrix} k_2^2 & -k_1 k_2 \\ -k_1 k_2 & k_1^2 \end{pmatrix} \begin{pmatrix} \widehat{f}_{1,k} \\ \widehat{f}_{2,k} \end{pmatrix} \tag{6.14}$$

for $k \neq 0$. In physical space we define P by $P = \mathcal{F}^{-1}\widehat{P}\mathcal{F}$.

Lemma 6.2.2. *The projection $\widehat{P}$ is a bounded linear map in $\ell_{1,m}$ and in $\ell_{2,m}$, i.e., for all $m \in \mathbb{R}$ there exists a $C > 0$ such that*

$$\|\widehat{P}\widehat{f}\|_{\ell_{1,m}} \leq C\|\widehat{f}\|_{\ell_{1,m}} \quad \textit{and} \quad \|\widehat{P}\widehat{f}\|_{\ell_{2,m}} \leq C\|\widehat{f}\|_{\ell_{2,m}}.$$

Hence, the projection P is also a bounded linear map in H^m_{per}, i.e., for all $m \in \mathbb{R}$ there exists a $C > 0$ such that

$$\|Pf\|_{H^m} \leq C\|f\|_{H^m}.$$

Proof. We find

$$\|\widehat{P}\widehat{f}\|_{\ell_{1,m}} = \|(\widehat{P}_k \widehat{f}_k)_{k\in\mathbb{Z}^2}\|_{\ell_{1,m}} \leq \sup_{k\in\mathbb{Z}^2} \|\widehat{P}_k\| \, \|(\widehat{f}_k)_{k\in\mathbb{Z}^2}\|_{\ell_{1,m}} \leq C\|\widehat{f}\|_{\ell_{1,m}}.$$

The proof for $\ell_{2,m}$ works exactly the same. Using that Fourier transform is an isomorphism between $\ell_{2,m}$ and H^m_{per}, cf. Lemma 5.2.9, yields

$$\|Pf\|_{H^m} \leq C_1\|\widehat{P}\widehat{f}\|_{\ell_{2,m}} \leq C_1 C\|\widehat{f}\|_{\ell_{2,m}} \leq C_1 C C_2\|f\|_{H^m}. \qquad \square$$

Remark 6.2.3. a) Lemma 6.2.2 is valid in arbitrary dimensions $d \geq 2$.
b) In Fourier space we have $\widehat{v}_k \in V_k := \{\widehat{v}_k \in \mathbb{C}^d : k \cdot \widehat{v}_k = 0\}$, due to $\nabla \cdot v = 0$. The pressure gradient ∇p defines in each $\mathbb{C}^d$ a vector $\mathrm{i}\widehat{p}_k k$, with $\widehat{p}_k \in \mathbb{C}$, which is orthogonal to V_k. This property can be generalized to general domains $\Omega \subset \mathbb{R}^d$, cf. §6.3. ⌋

The phase space and the fixed point argument. In the following we will solve the Fourier transformed Navier-Stokes equations in two classes of phase spaces, namely

$$\ell^{div}_{1,m} = \{\widehat{u} \in (\ell_{1,m})^d : \widehat{u} = \widehat{P}\widehat{u}\}, \qquad \ell^{div}_{2,m} = \{\widehat{u} \in (\ell_{2,m})^d : \widehat{u} = \widehat{P}\widehat{u}\},$$

and the Navier-Stokes equations in physical space in the class of phase spaces

$$H^{div,m}_{\text{per}} = \{u \in (H^m_{\text{per}})^d : u = Pu\}.$$

In case of periodic boundary conditions we have $P\Delta = \Delta P$. Hence, for u with $u = Pu$ the solution operator of the linearized Navier-Stokes equations is given coordinate-wise by the solution operator $\mathrm{e}^{t\Delta}$ of the linear diffusion equation $\partial_t u = \Delta u$. In one space dimension this operator has been discussed a number of times, cf. Example 5.1.21 and Example 5.2.19. The statements made above about this operator transfer line to line from $\mathbb{R}^1$ to $\mathbb{R}^d$.

Theorem 6.2.4. *The solution operator* $(\mathrm{e}^{-|k|^2 t})_{k\in\mathbb{Z}^d}$ *of the linearized Navier-Stokes equations* $\partial_t \widehat{u}_k = -|k|^2 \widehat{u}_k$ *in Fourier space defines a* C_0*-semigroup in* $\ell^{div}_{1,m}$ *and* $\ell^{div}_{2,m}$ *for all* $m \in \mathbb{R}$. *The associated solution operator* $\mathrm{e}^{t\Delta} = \mathcal{F}^{-1}(\mathrm{e}^{-|k|^2 t})_{k\in\mathbb{Z}^d}\mathcal{F}$ *of the linearized Navier-Stokes equations in physical space defines a* C_0*-semigroup in* $H^{div,m}_{\mathrm{per}}$. *Moreover, for all* $m \in \mathbb{R}$ *and* $r \geq 0$ *there exists a* $C > 0$, *such that*

$$\begin{aligned}
\|(\mathrm{e}^{-|k|^2 t})_{k\in\mathbb{Z}^d}\widehat{u}\|_{\ell^{div}_{1,m+r}} &\leq C \max(1, t^{-r/2})\|\widehat{u}\|_{\ell^{div}_{1,m}}, \\
\|(\mathrm{e}^{-|k|^2 t})_{k\in\mathbb{Z}^d}\widehat{u}\|_{\ell^{div}_{2,m+r}} &\leq C \max(1, t^{-r/2})\|\widehat{u}\|_{\ell^{div}_{2,m}}, \\
\|\mathrm{e}^{t\Delta} u\|_{H^{div,m+r}_{\mathrm{per}}} &\leq C \max(1, t^{-r/2})\|u\|_{H^{div,m}_{\mathrm{per}}}.
\end{aligned}$$

Since we already analyzed P in §6.2.1 it remains to bound the nonlinear term $(u\cdot\nabla)u)$. It is easy to see that it is smooth from $H^{div,m}_{\mathrm{per}}$ to H^{m-1}_{per} if m is sufficiently large. Hence, with the previous estimates the local existence and uniqueness of solutions easily follows in every $H^{div,m}_{\mathrm{per}}$ if m is sufficiently large. However, since we are also interested in the global existence and uniqueness of solutions, and since it is more easy to obtain a priori estimates on the solutions in $H^{div,m}_{\mathrm{per}}$-spaces for small m, we would like to have m as small as possible. Hence, we spend a little bit more time at this point to optimize the estimates.

We use the variation of constant formula (6.11) to prove local existence and uniqueness of solutions. The key ingredients are the smoothing properties of the semigroup $\mathrm{e}^{t\Delta}$ summarized in Theorem 6.2.4 and the Lipschitz-continuity of the nonlinear terms $P(u\cdot\nabla)u)$. Since a singularity $t^{-1+\delta}$, with $\delta > 0$, is integrable, our approach also works if the nonlinearity $P((u\cdot\nabla)u)$ is Lipschitz-continuous from $H^{div,m}_{\mathrm{per}}$ into $H^{div,m-2+\delta}_{\mathrm{per}}$. By using the incompressibility $\sum_{j=1}^d \partial_{x_j} u_j = 0$ the l^{th} component of $(u\cdot\nabla)u$ can be written as

$$\sum_{j=1}^d u_j \partial_{x_j} u_l = \sum_{j=1}^d u_j \partial_{x_j} u_l + \sum_{j=1}^d u_l \partial_{x_j} u_j = \sum_{j=1}^d \partial_{x_j}(u_j u_l),$$

or equivalently in vector notation as

$$(u\cdot\nabla)u = \nabla\cdot(uu^T). \tag{6.15}$$

Using this representation we have to establish

$$\|\nabla\cdot uv^T\|_{H^{m-2+\delta}} \leq C\|u\|_{H^m}\|v\|_{H^m},$$

or equivalently

$$\|uv^T\|_{H^{m-1+\delta}} \leq C\|u\|_{H^m}\|v\|_{H^m} \tag{6.16}$$

for a $\delta > 0$ and all $m > m^*$ for a $m^* \in \mathbb{R}$. To find the minimal m^* we start with the following lemma.

Lemma 6.2.5. *For all $\widetilde{m} \in \mathbb{R}$, $r \geq 0$, and $\delta > 0$ there exists a $C > 0$ such that*

$$\|\widehat{u} * \widehat{v}\|_{\ell_{2,\widetilde{m}}} \leq C \left(\|\widehat{u}\|_{\ell_{2,\widetilde{m}+r}}\|\widehat{v}\|_{\ell_{2,\frac{d}{2}-r+\delta}} + \|\widehat{u}\|_{\ell_{2,\frac{d}{2}-r+\delta}}\|\widehat{v}\|_{\ell_{2,\widetilde{m}+r}}\right).$$

Proof. We define $\widehat{\varrho}(k) = (1 + |k|)^{\widetilde{m}}$, $\widehat{\varrho}_1(k) = (1 + |k|)^{\widetilde{m}+r}$, and $\widehat{\varrho}_2(k) = (1 + |k|)^{-r}$, for which we have the inequality

$$\begin{aligned} \widehat{\varrho}(k) \leq& (1+|k|)^{\widetilde{m}} \leq C((1+|k-l|)^{\widetilde{m}+r}(1+|l|)^{-r}+(1+|l|)^{\widetilde{m}+r}(1+|k-l|)^{-r}) \\ \leq& C(\widehat{\varrho}_1(k-\ell)\widehat{\varrho}_2(\ell) + \widehat{\varrho}_2(k-\ell)\widehat{\varrho}_1(\ell)). \end{aligned}$$

Using this and Lemma 5.1.26 yields

$$\begin{aligned} \|\widehat{u} * \widehat{v}\|_{\ell_{2,\widetilde{m}}} =& \|\widehat{\varrho}(\widehat{u} * \widehat{v})\|_{\ell_2} = 2\| \left(\widehat{u}\widehat{\varrho}_1\right) * \left(\widehat{v}\widehat{\varrho}_2\right) \|_{\ell_2} + 2\| \left(\widehat{u}\widehat{\varrho}_2\right) * \left(\widehat{v}\widehat{\varrho}_1\right) \|_{\ell_2} \\ \leq& 2\|\widehat{u}\widehat{\varrho}_1\|_{\ell_2}\|\widehat{v}\widehat{\varrho}_2\|_{\ell_1} + 2\|\widehat{u}\widehat{\varrho}_2\|_{\ell_1}\|\widehat{v}\widehat{\varrho}_1\|_{\ell_2} \\ =& 2C\left(\|\widehat{u}\|_{\ell_{2,\widetilde{m}+r}}\|\widehat{v}\|_{\ell_{1,-r}} + 2\|\widehat{u}\|_{\ell_{1,-r}}\|\widehat{v}\|_{\ell_{2,\widetilde{m}+r}}\right) \\ \leq& 2C\widetilde{C}\left(\|\widehat{u}\|_{\ell_{2,\widetilde{m}+r}}\|\widehat{v}\|_{\ell_{2,\frac{d}{2}-r+\delta}} + \|\widehat{u}\|_{\ell_{2,\frac{d}{2}-r+\delta}}\|\widehat{v}\|_{\ell_{2,\widetilde{m}+r}}\right) \end{aligned}$$

for a $\delta > 0$ according to Sobolev's embedding, cf. Lemma 5.1.27. □

For the validity of (6.16) we have to choose $r = 1 - \delta$ for a $\delta > 0$ such that $m^* = d/2-1$ and use that Fourier transform is an isomorphism between $\ell_{2,m}$ and H^m_{per}.

Remark 6.2.6. (The critical Sobolev number m^* for $\ell_{1,m}$) For all $m \geq 0$ there exists a constant C such that $(1+|k+l|)^m \leq C((1+|k|)^m+(1+|l|)^m)$ for all k and l. Thus, we have $\|\widehat{u} * \widehat{v}\|_{\ell_{1,m}} \leq C\|\widehat{u}\|_{\ell_{1,m}}\|\widehat{v}\|_{\ell_{1,m}}$ which yields $m \geq m^* = 0$. ⌋

Therefore, we obtain the following local existence and uniqueness result.

Theorem 6.2.7. *a) Let $\widehat{u}_0 \in \ell^{div}_{2,m}$ for $m > m^* = d/2-1$. Then there exists a $T_0 = T_0(\|\widehat{u}_0\|_{\ell^{div}_{2,m}}) > 0$ and unique mild solution $\widehat{u} \in C([0,T_0], \ell^{div}_{2,m})$ of the Fourier transformed Navier-Stokes equations (6.11) with $\widehat{u}|_{t=0} = \widehat{u}_0$. The same is true if $\ell^{div}_{2,m}$ is replaced by $\ell^{div}_{1,m}$ and $m \geq 0$.*

b) Let $u_0 \in H^{div,m}_{\text{per}}$ for $m > m^* = d/2-1$. Then there exists a $T_0 = T_0(\|u_0\|_{H^{div,m}_{\text{per}}}) > 0$ and unique mild solution $u \in C([0,T_0], H^{div,m}_{\text{per}})$ of the Navier-Stokes equations (6.11) with $u|_{t=0} = u_0$.

6.2.2. Analyticity of solutions. As pointed out in §5.3.3 the solutions of semi-linear equations where the semigroup is smoothing are infinitely often differentiable for every $t > 0$. The scheme which has been explained in §5.3.3 can also be applied to the Navier-Stokes equations if $m > m^*$. However, the step size for increasing the differentiability has to be decreased from 1 to $\delta/2$. Actually, the solution becomes analytic in a strip along the real axis in the complex plane if $m > m^*$. In order to prove this we define

Definition 6.2.8. *For $\beta \geq 0$ let*

$$\ell^\omega_{1,\beta} = \{\widehat{u} : \mathbb{Z}^d \to \mathbb{C} : \|\widehat{u}\|_{\ell^\omega_{1,\beta}} < \infty\} \quad \textit{and} \quad \|\widehat{u}\|_{\ell^\omega_{1,\beta}} = \sum_{k\in\mathbb{Z}^d} |\widehat{u}_k| \mathrm{e}^{\beta|k|}.$$

Lemma 6.2.9. *If $\widehat{u} \in \ell^\omega_{1,\beta}$ then $u = \mathcal{F}^{-1}\widehat{u}$ is analytic in a strip*

$$\mathcal{S}_\beta = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \max_{j=1,\ldots,d} |\mathrm{Im} z_j| < \beta\}$$

in $\mathbb{C}^d$ and $\sup_{z\in\mathcal{S}_\beta} |u(z)| \leq \|\widehat{u}\|_{\ell^\omega_{1,\beta}}$.

Proof. The estimate

$$\sup_{z\in\mathcal{S}_\beta} |u(z)| \leq \sum_{k\in\mathbb{Z}^d} (|\widehat{u}_k| \sup_{z\in\mathcal{S}_\beta} |\mathrm{e}^{ikz}|) \leq \sum_{k\in\mathbb{Z}^d} (|\widehat{u}_k| \mathrm{e}^{\beta|k|}) \leq \|\widehat{u}\|_{\ell^\omega_{1,\beta}} < \infty \tag{6.17}$$

shows that the function $u(z) = \sum_{k\in\mathbb{Z}^d} \widehat{u}_k \mathrm{e}^{ikz}$ is the uniform limit of the analytic functions $z \mapsto s_N(z) = \sum_{|k|\leq N} \widehat{u}_k \mathrm{e}^{ikz}$ in their domain of convergence $\mathcal{S}_\beta$ and so the function u is analytic in $\mathcal{S}_\beta$. □

For every $t > 0$ and $\beta \geq 0$ the linear Navier-Stokes semigroup $\widehat{T}(t)$ defined by $(\widehat{T}(t)\widehat{u})_k = \mathrm{e}^{-|k|^2 t}\widehat{u}_k$ maps ℓ_1 into $\ell^\omega_{1,\beta}$ due to

$$\|\widehat{T}(t)\widehat{u}\|_{\ell^\omega_{1,\beta}} \leq \sum_{k\in\mathbb{Z}^d} |\mathrm{e}^{-|k|^2 t}\widehat{u}_k \mathrm{e}^{\beta|k|}| \leq \sup_{k\in\mathbb{Z}^d} |\mathrm{e}^{-|k|^2 t}\mathrm{e}^{\beta|k|}| \sum_{k\in\mathbb{Z}^d} |\widehat{u}_k| \leq C(\beta, t)\|\widehat{u}\|_{\ell_1},$$

where $\sup_{k\in\mathbb{Z}^d} |\mathrm{e}^{-|k|^2 t}\mathrm{e}^{\beta|k|}| \leq C(\beta, t) < \infty$, since $\mathrm{e}^{-|k|^2 t}\mathrm{e}^{\beta|k|} \to 0$ for $|k| \to \infty$ if $t > 0$. The constant $C = C(\beta, t)$ satisfies $C(\beta, t) \to \infty$ for $t \to 0$ with a non-integrable singularity which makes this estimate useless for nonlinear problems. However, this singularity can be avoided if we choose β proportional to $\sqrt{t}$.

Lemma 6.2.10. *There exists a constant $C < \infty$ such that the semigroup $\widehat{T}(t)$ defined by $(\widehat{T}(t)\widehat{u})_k = \mathrm{e}^{-|k|^2 t}\widehat{u}_k$ satisfies for all $t \geq 0$ that*

$$\|\widehat{T}(t)\widehat{u}\|_{\ell^\omega_{1,\sqrt{t}}} \leq C\|\widehat{u}\|_{\ell_1}.$$

Proof. The assertion follows since

$$\sum_{k\in\mathbb{Z}^d} |\mathrm{e}^{-|k|^2 t}\mathrm{e}^{\sqrt{t}|k|}| \leq \sum_{s\in\mathbb{R}^d} |\mathrm{e}^{-|s|^2}\mathrm{e}^{|s|}| \leq C < \infty$$

can be bounded independently of t. □

This estimate can be used to prove the analyticity of the solutions of Navier-Stokes equations w.r.t. x in $\mathcal{S}_{\sqrt{t}} \subset \mathbb{C}$ for $t > 0$. According to Lemma 6.2.9 this assertion follows if $\widehat{u}(t) \in \ell^{\omega}_{1,\sqrt{t}}$ for $t > 0$.

Lemma 6.2.11. *For every $M > 0$ there exists a $T_0 > 0$ such that the map*

$$F(\widehat{u})(k,t) = \mathrm{e}^{-k^2 t}\widehat{u}(k,0) + \int_0^t \mathrm{e}^{-k^2(t-\tau)}\widehat{P}(k)\mathrm{i}k(\widehat{u} * \widehat{u}^T)(k,\tau)\,\mathrm{d}\tau$$

is a contraction in

$$\mathcal{X} = \{\widehat{u} : \mathbb{Z} \times [0,T_0] \to \mathbb{C} : \|\widehat{u} - \widehat{u}_{lin}\|_{\mathcal{X}} \le M\},$$

where $\|\widehat{u}\|_{\mathcal{X}} = \sup_{t\in[0,T_0]} \|\widehat{u}(t)\|_{\ell^{\omega}_{1,\sqrt{t}}}$ *and where* $\widehat{u}_{lin}(k,t) = \mathrm{e}^{-k^2 t}\widehat{u}(k,0)$.

Proof. It is easy to see, cf. Exercise 6.4, that $\|\widehat{u} * \widehat{v}\|_{\ell^{\omega}_{1,\sqrt{t}}} \le \|\widehat{u}\|_{\ell^{\omega}_{1,\sqrt{t}}}\|\widehat{v}\|_{\ell^{\omega}_{1,\sqrt{t}}}$ which implies

$$\|u * v\|_{\mathcal{X}} \le \|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}. \tag{6.18}$$

Using this estimate and $\sqrt{t-s} \ge \sqrt{t} - \sqrt{s}$ shows that

$$\begin{aligned}
\|F(\widehat{u}) - \widehat{u}_{lin}\|_{\mathcal{X}} &\le \sup_{t\in[0,T_0]} \sum_{k\in\mathbb{Z}} \Big| \int_0^t \mathrm{e}^{-k^2(t-\tau)}\widehat{P}(k)\mathrm{i}k(\widehat{u} * \widehat{u}^T)(k,\tau)\,\mathrm{d}\tau \Big| \mathrm{e}^{\sqrt{t}|k|} \\
&\le \sup_{t\in[0,T_0]} \sum_{k\in\mathbb{Z}} \int_0^t |\mathrm{e}^{-k^2(t-\tau)}\widehat{P}(k)\mathrm{i}k(\widehat{u} * \widehat{u}^T)(k,\tau)\mathrm{e}^{\sqrt{t}|k|}\mathrm{e}^{-\sqrt{\tau}|k|}\mathrm{e}^{\sqrt{\tau}|k|}|\,\mathrm{d}\tau \\
&\le \sup_{t\in[0,T_0]} \int_0^t \sum_{k\in\mathbb{Z}} |\mathrm{e}^{-k^2(t-\tau)}\widehat{P}(k)\mathrm{i}k\mathrm{e}^{\sqrt{t-\tau}|k|}(\widehat{u} * \widehat{u}^T)(k,\tau)\mathrm{e}^{\sqrt{\tau}|k|}|\,\mathrm{d}\tau \\
&\le C \sup_{t\in[0,T_0]} \int_0^t \sup_{k\in\mathbb{Z}} |\mathrm{e}^{-k^2(t-\tau)}\mathrm{i}k\mathrm{e}^{\sqrt{t-\tau}|k|}|\,\mathrm{d}\tau \sup_{s\in[0,T_0]} \sum_{k\in\mathbb{Z}} |(\widehat{u} * \widehat{u}^T)(k,\tau)\mathrm{e}^{\sqrt{\tau}|k|}| \\
&\le C T_0^{1/2}\|\widehat{u}\|_{\mathcal{X}}^2 < \infty.
\end{aligned}$$

Thus, we proved that F maps the space $\mathcal{X}$ into itself if $T_0 > 0$ is sufficiently small. The proof of the contraction property works the same way. □

Corollary 6.2.12. *For all $C > 0$ there exists a $T_0 > 0$ such that the solutions u of the Navier-Stokes equations are analytic w.r.t. x in $\mathcal{S}_{\sqrt{t}} \subset \mathbb{C}$ for all $t \in [0,T_0]$ if $\|\widehat{u}(0)\|_{\ell_1} \le C$.*

Remark 6.2.13. It is easy to see that Theorem 5.1.23 can be generalized to $\ell^{\omega}_{1,\beta}$-spaces and the Fourier transformed Navier-Stokes equations such that there is local existence and uniqueness in $\ell^{\omega}_{1,\beta}$, too. ⌋

Remark 6.2.14. In general it cannot be expected that in nonlinear problems the strip of analyticity is arbitrarily wide. An explicit, but typical, example is the $z \mapsto \tanh(z)$ equilibrium of the Allen-Cahn equation

$\partial_t u = \partial_x^2 u + u - u^3$ in the subsequent §7.2. The function tanh has singularities in the complex plane, due to $\tanh(iy) = \tan(y)$, for $z = i\pi/2 + ik\pi$ with $k \in \mathbb{Z}$. ⌋

6.2.3. Global existence in 2D. In order to prove the global existence and uniqueness of solutions of the Navier-Stokes equations in a phase space X we need a local existence and uniqueness result in X and a priori bounds for the solutions in X. Then as explained already a number of times the local existence and uniqueness result can be applied again and again to construct a solution for all $t \geq 0$.

Bounds for the L^2-norm. The L^2-norm of the solutions u of the Navier-Stokes equations can be bound in every space dimension. By using integration by parts and the incompressibility $\sum_{j=1}^d \partial_j u_j = 0$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\sum_{j=1}^d u_j u_j \,\mathrm{d}x &= \int_{\mathbb{T}^d}\sum_{j=1}^d u_j \partial_t u_j \,\mathrm{d}x \\ &= \int_{\mathbb{T}^d}\sum_{j=1}^d\sum_{\ell=1}^d u_j(\partial_{x_\ell}\partial_{x_\ell}u_j - \partial_j p - \sum_{\ell=1}^d u_\ell \partial_{x_\ell} u_j)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d} -(\partial_{x_\ell}u_j)(\partial_{x_\ell}u_j)\,\mathrm{d}x.\end{aligned}$$

Poincaré's inequality, cf. Lemma 5.2.17, implies

Lemma 6.2.15. *For all $d \geq 2$ we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 \leq -\|u\|_{L^2}^2, \quad \textit{and so} \quad \|u(t)\|_{L^2} \leq \mathrm{e}^{-t}\|u(0)\|_{L^2}\ .$$

Bounds for the H^1-norm. In $\mathbb{R}^2$ with periodic boundary conditions also the H^1-norm can be bound. By using integration by parts and the incompressibility $\sum_{j=1}^2 \partial_j u_j = 0$, in $\mathbb{R}^2$ we find after some explicit calculation, cf. Exercise 6.5, that

$$\int_{\mathbb{T}^2}\sum_{j=1}^2\sum_{\ell=1}^2\sum_{m=1}^2(\partial_{x_j}u_\ell)\partial_{x_j}(u_m\partial_{x_m}u_\ell) = 0. \tag{6.19}$$

Therefore, we find as above

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\sum_{j=1}^2\sum_{\ell=1}^2(\partial_{x_j}u_\ell)(\partial_{x_j}u_\ell)\,\mathrm{d}x = -\int_{\mathbb{T}^2}\sum_{j=1}^2\sum_{k=1}^2\sum_{\ell=1}^2(\partial_{x_j}\partial_{x_k}u_\ell)(\partial_{x_j}\partial_{x_k}u_\ell)\,\mathrm{d}x.$$

Again with Poincaré's inequality, cf. Lemma 5.2.17, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\sum_{j=1}^2\sum_{\ell=1}^2(\partial_{x_j}u_\ell)(\partial_{x_j}u_\ell)\,\mathrm{d}x = -\int_{\mathbb{T}^2}\sum_{j=1}^2\sum_{\ell=1}^2(\partial_{x_j}u_\ell)(\partial_{x_j}u_\ell)\,\mathrm{d}x.$$

Combining this estimate with the L^2-estimate yields

Lemma 6.2.16. *For $d = 2$ (and only for $d = 2$ and periodic boundary conditions) we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^1}^2 \le -\|u(t)\|_{H^1}^2 \quad \textit{and so} \quad \|u(t)\|_{H^1} \le \mathrm{e}^{-t}\|u(0)\|_{H^1}.$$

Combining this a priori estimate with the previous local existence and uniqueness result in $\mathbb{R}^2$ yields

Theorem 6.2.17. *Let $u_0 \in H_{\mathrm{per}}^{div,1}(\mathbb{T}^2)$. Then there exists a unique mild solution $u \in C([0,\infty), H_{\mathrm{per}}^{div,1}(\mathbb{T}^2))$ of the Navier-Stokes equations (6.11) with $u|_{t=0} = u_0$. Moreover, for $t > 0$ the solution is an analytic function.*

The same is true for every $H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)$ with $m > m^* = 0$. For $m \in (m^*, 1)$ the result follows from local existence and uniqueness of solutions in such $H_{\mathrm{per}}^{div,m}(\mathbb{R}^2)$ and due to $\|u(t)\|_{H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)} \le \|u(t)\|_{H_{\mathrm{per}}^{div,1}(\mathbb{T}^2)}$ for such m. For $m > 1$ we have

$$\|uv^T\|_{H^m} \le C\|u\|_{H^m}\|v\|_{H^m} \tag{6.20}$$

and the local existence and uniqueness in $H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)$. Arguing as in §5.3.3 yields an a priori-bound for $\|u(t)\|_{H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)}$ in terms of $\|u(t-T_0)\|_{H_{\mathrm{per}}^{div,1}(\mathbb{T}^2)}$ for all $t \ge T_0$. Since $\|u(t-T_0)\|_{H_{\mathrm{per}}^{div,1}(\mathbb{T}^2)}$ is globally bounded and decays to zero, the local existence and uniqueness result in $H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)$ can be applied again and again to construct a solution for all $t \ge 0$. Therefore, we have the following theorem about the asymptotic stability of the zero solution.

Theorem 6.2.18. *Let $u_0 \in H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)$ with $m > m^* = 0$. Then there exists a unique mild solution $u \in C([0,\infty), H_{\mathrm{per}}^{div,m}(\mathbb{T}^2))$ of the Navier-Stokes equations (6.11) with $u|_{t=0} = u_0$. Moreover, for $t > 0$ the solution is an analytic function and satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{H_{\mathrm{per}}^{div,m}(\mathbb{T}^2)} = 0$$

with some exponential rate.

6.2.4. The millennium problem. Figure 6.4 displays the Sobolev numbers for the local existence and uniqueness and the a priori estimates.

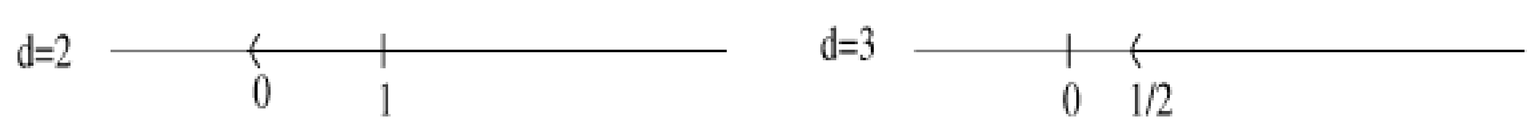


Figure 6.4. A priori estimates in L^2 for $d = 3$ and H^1 for $d = 2$. Local existence and uniqueness holds in H^m for $m > d/2 - 1$. For $d = 2$ there is no gap between the exponents m for which we have a priori estimates and the ones for which we have local existence and uniqueness of solutions. For $d = 3$ a gap remains, and global existence of smooth solutions cannot be concluded.

In $\mathbb{R}^3$ we have no a priori estimate for the spaces where we have local existence and uniqueness. Hence, global existence of smooth solutions is unclear. This question is exactly the content of the millennium problem formulated in [**Fef06**]:

Millennium problem of the Clay-foundation. *Prove (or disprove) the global existence and uniqueness of solutions of the Navier-Stokes equations in three space dimensions. For instance, show $T_0 = \infty$ in Theorem* 6.2.7, *i.e., close the gap between the a priori estimates and the local existence- and uniqueness theorem, respectively.*

Partial results are already known, from which we list only the two absolute basic ones.

- Jean Leray [**Ler34**] proved the global existence of so called weak solutions, cf. §7.4.2, of the Navier-Stokes equations. These solutions are very rough and they are not unique.
- For small initial conditions due to the linear stability of the origin one easily obtains an a priori estimate and thus it follows $\lim_{t\to\infty} \|u(t)\|_{H^{div,m}_{\text{per}}(\mathbb{T}^3)} = 0$ with some exponential rate for every $m > m^* = 1/2$.

6.2.5. Some qualitative theory. The 2D Navier-Stokes equations without forcing are a bit boring since $\|u(t)\|_{H^1} \to 0$ for $t \to \infty$, cf. Lemma 6.2.16. Thus, we show the existence of a global attractor for the two-dimensional Navier-Stokes equations with forcing, i.e.,

$$\partial_t u = \Delta u - \nabla p - (u \cdot \nabla)u + f, \quad \nabla \cdot u = 0, \tag{6.21}$$

with 2π-periodic boundary conditions and external (time-independent) force $f \in L^2$ with $\widehat{f}_0 = 0$. As above we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^1}^2 \le -\|u\|_{H^1}^2 + \|u\|_{L^2}\|f\|_{L^2} \le -\frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}\|f\|_{L^2}^2$$

using $\|u\|_{L^2}\|f\|_{L^2} \le \frac{1}{2}\|u\|_{L^2}^2 + \frac{1}{2}\|f\|_{L^2}^2$, and so

$$\|u(t)\|_{H^1} \le \mathrm{e}^{-t}\|u_0\|_{H^1} + \|f\|_{L^2}(1 - \mathrm{e}^{-t}).$$

Hence, the set

$$\mathcal{B} = \{u \in H^1 : \|u\|_{H^1} \le 2\|f\|_{H^1}\}$$

is absorbing, i.e., attracts balls of finite size in finite times. Moreover, it is (positively) invariant under the flow of (6.21), i.e., $u_0 \in \mathcal{B}$ implies $u(t, u_0) \in \mathcal{B}$ for all $t \ge 0$.

Since the embedding $H^1 \hookrightarrow L^2$ is compact (cf. Theorem 5.1.33), as in Theorem 5.3.4 we thus obtain the existence of the global attractor (in L^2), given by

$$\mathcal{A} = \bigcap_{t \geq 0} A_t, \qquad \text{with} \quad A_t = \overline{\mathcal{S}_t(\mathcal{B})}.$$

Theorem 6.2.19. *The 2D Navier-Stokes equations* (6.21) *have a non-empty, compact, time-invariant set $\mathcal{A} \subset L^2$, the global attractor, with*

$$\operatorname{dist}_{L^2}(S_t(\mathcal{B}), \mathcal{A}) = \sup_{b \in S_t(\mathcal{B})} \inf_{a \in \mathcal{A}} \|a - b\|_{L^2} \to 0 \ \textit{as}\ t \to \infty.$$

Remark 6.2.20. With a little more work it can be shown that $\mathcal{A}$ is a H^1_{per} attractor for (6.21), i.e., $\mathcal{A}$ is compact in H^1 and attracts in H^1, i.e.,

$$\operatorname{dist}_{H^1}(S_t(\mathcal{B}), \mathcal{A}) = \sup_{b \in S_t(\mathcal{B})} \inf_{a \in \mathcal{A}} \|a - b\|_{H^1} \to 0 \text{ as } t \to \infty.$$

To show this, use a priori estimates to obtain an absorbing set in H^2_{per}. ⌋

6.3. Other boundary conditions and more general domains

In this section we consider the Navier-Stokes equations with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$ in an open domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. In order to prove the local existence and uniqueness of solutions we generalize our previous approach and recall the basics of analytic semigroup theory, cf. [**Hen81**]. We recall that the resolvent set of a (bounded or unbounded) linear operator $A : D(A) \subset X \to X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which $(\lambda - A)$ has a bounded inverse $(\lambda - A)^{-1} : X \to X$, the resolvent.

Definition 6.3.1. *A closed and densely defined operator A in a Banach space X is called sectorial if there exists an $a \in \mathbb{R}$, a $\phi \in (0, \frac{\pi}{2})$ and an $M \geq 1$, such that the sector*

$$S_{a,\phi} = \{\ \lambda : \phi \leq |\arg(\lambda - a)| \leq \pi,\ \lambda \neq a\ \}$$

is a part of the resolvent set of A, and such that for all $\lambda \in S_{a,\phi}$ we have the estimate

$$\|(\lambda - A)^{-1}\|_{X \to X} \leq \frac{M}{|\lambda - a|}.$$

The negative of a sectorial operator generates an analytic semigroup.

Definition 6.3.2. *A C_0-semigroup $T(t)$ of bounded linear operators is called analytic if $t \mapsto T(t)u$ is analytic for $0 < t < \infty$ and all $u \in X$.*

The following theorem gives an explicit construction of the semigroup generated by the negative of a sectorial operator.

Theorem 6.3.3. *Let A be a sectorial operator. Then $-A$ generates an analytic semigroup with the representation*

$$\mathrm{e}^{-tA} := \frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1}\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda,$$

where Γ is a curve in the resolvent set $\rho(-A)$ with $\arg\lambda \to \pm\theta$ for $|\lambda| \to \infty$ and a $\theta \in (\frac{\pi}{2}, \pi)$. The semigroup can be extended analytically into the sector $\{t \neq 0 : |\arg t| \leq \varepsilon\}$ for a $\varepsilon > 0$. If $\operatorname{Re}\lambda > a$ for λ in the spectrum $\sigma(A)$ then

$$\|\mathrm{e}^{-tA}\| \leq C\mathrm{e}^{-at} \qquad \textit{und} \qquad \|A\mathrm{e}^{-tA}\| \leq \frac{C}{t}\mathrm{e}^{-at}$$

for $t > 0$ and a constant C. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-tA} = -A\mathrm{e}^{-tA}.$$

Remark 6.3.4. a) For $t > 0$ the integral

$$\frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1}\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda$$

is well defined since $\operatorname{Re}\lambda \to -\infty$ for $|\lambda| \to \infty$ and $\|(\lambda + A)^{-1}\|_{X\to X} \leq \frac{M}{|\lambda - a|}$. Since the resolvent is holomorphic on the resolvent set, due to Cauchy's theorem of complex analysis the value of the integral is independent of the special choice of Γ.

b) The estimate $\|A\mathrm{e}^{-tA}\|_{X\to X} \leq \frac{C}{t}\mathrm{e}^{-at}$ implies that e^{-tA} maps the space X into the domain of definition $D(A)$. Since $A^n\mathrm{e}^{-tA} = (A\mathrm{e}^{-A\frac{t}{n}})^n$ we even have

$$\|A^n\mathrm{e}^{-tA}\|_{X\to X} \leq \|A\mathrm{e}^{-A\frac{t}{n}}\|^n_{X\to X} \leq \left(\frac{C}{t}\mathrm{e}^{-a\frac{t}{n}}\right)^n \leq \frac{\widetilde{C}}{t^n}\mathrm{e}^{-at},$$

such that $\mathrm{e}^{-tA} : X \to D(A^n)$ for $t > 0$, and the semigroup is smoothing. ⌋

Remark 6.3.5. a) The concept of sectorial operators is very robust under perturbations. Let A be a sectorial operator with $\|A(\lambda - A)^{-1}\| \leq C$ for all λ in a chosen sector. Moreover, let B be a linear operator with $D(B) \supset D(A)$ satisfying

$$\|Bx\| \leq \varepsilon\|Ax\| + K\|x\|$$

with ε, K some constants. If $\varepsilon C < 1$, then also $A + B$ is a sectorial operator. For a self-adjoint A it is sufficient that $\varepsilon < 1$. Hence, it is sufficient to check the assumptions for the principal part of a given operator. Such estimates can often be found in the existing literature.

b) The most essential remark is that, due to Parseval's identity, in a Hilbert space every self-adjoint operator which is bounded from below is a sectorial operator. ⌋

In order to apply the previous ideas to the Navier-Stokes equations we first have to get rid of the pressure term and of the equation $\nabla \cdot u = 0$. Therefore, let $u \in C^1(\Omega, \mathbb{R}^d)$ with $\nabla \cdot u = 0$ and $u \cdot n|_{\partial\Omega} = 0$. Then for $\phi \in C^1(\Omega, \mathbb{R})$ we have $\int_\Omega u \cdot \nabla\phi \, \mathrm{d}x = 0$. On the other hand a vector field u which is orthogonal to $\{\nabla\phi : \phi \in C^1(\Omega, \mathbb{R})\}$ satisfies $\nabla \cdot u = 0$ and $u \cdot n|_{\partial\Omega} = 0$.

We define L^2_p to be the closure of $\{\nabla\phi : \phi \in C^1(\Omega, \mathbb{R})\}$ and L^2_{div} to be the closure of $\{u \in C^1(\Omega, \mathbb{R}^d) : \nabla \cdot u = 0, \quad u \cdot n|_{\partial\Omega} = 0\}$. Then L^2_p and L^2_{div} are orthogonal subspaces of L^2 with $L^2 = L^2_P \oplus L^2_{div}$. We introduce P to be the orthogonal projection on the subspace L^2_{div}.

As before we write the Navier Stokes equations as $\partial_t u = -Au + N(u)$ with $Au = -P\Delta$ under Dirichlet boundary conditions and $N(u) = -P((u \cdot \nabla)u)$. It is easy to see that A is a self adjoint and positive definite operator which immediately implies that A is a sectorial operator, too. Hence, $-A$ is the generator of an analytic semigroup, cf. Remark 6.3.5 b).

Again u is called mild solution if u satisfies the variation of constant formula

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-\tau)A}N(u(\tau)) \, \mathrm{d}\tau.$$

In order to control the nonlinear terms we need so called X^α-spaces. We consider a sectorial operator A with $\operatorname{Re}\sigma(A) > \delta > 0$. For a given sectorial operator A this can always be achieved by considering $\widetilde{A} = A + \beta I$ for a suitable chosen $\beta > 0$.

Definition 6.3.6. *For $\alpha > 0$ define*

$$A^{-\alpha} = -\frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda I - A)^{-1} \, \mathrm{d}\lambda,$$

where Γ is a curve asymptotically coming from $\mathrm{e}^{-i\theta}\infty$ and asymptotically going to $\mathrm{e}^{i\theta}\infty$ with $\frac{\pi}{2} - \widetilde{\delta} < \theta < \pi$ running between the origin and $\sigma(A)$. The branch of the function $\lambda \mapsto \lambda^{-\alpha}$ is chosen in such a way that the slit in the complex plane where $\lambda \mapsto \lambda^{-\alpha}$ is not analytic coincides with the negative real axis, see [**Gam01**] *for an introduction to complex analysis.*

Since $\|(\lambda I - A)^{-1}\| \leq \frac{C}{|\lambda|}$ we have the convergence of the integral for $\alpha > 0$. There exists a $C \geq 0$, such that $\|A^{-\alpha}\| \leq C$ for $0 < \alpha \leq 1$. Moreover, we have $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ if $\alpha, \beta \in (0, 1]$.

Since $\operatorname{Re}\sigma(A) > \delta > 0$ we have the injectivity of A^{-n}. Since $A^{-n} = A^{-n+\alpha}A^{-\alpha}$ for $n > \alpha$ we have the injectivity of $A^{-\alpha}$. Therefore, $A^{-\alpha} : X \to \mathbb{R}(A^{-\alpha})$ is bijective and we have a (non bounded) inverse. Other representation formulas for A^α can be found for instance in [**Hen81**, §1.4]

Definition 6.3.7. *We set $A^\alpha = (A^{-\alpha})^{-1}$ for $\alpha > 0$. The domain of definition is given by $D(A^\alpha) = R(A^{-\alpha})$. We introduce $X^\alpha = D(A^\alpha)$ equipped with the norm*

$$\|u\|_{X^\alpha} = \|A^\alpha u\|_X.$$

For $\alpha \in (0,1)$ we have $D(A) \subset D(A^\alpha)$ since $R(A^{-1}) \subset R(A^{-\alpha})$, and so $D(A^\alpha)$ is dense in X. Due to the construction of the operators we have that

$$\|e^{-tA}u\|_{X^\alpha} \le M_\alpha t^{-\alpha} e^{-\delta t}\|u\|_X.$$

In order to proceed with the local existence uniqueness of solutions as above we need the Lipschitz-continuity of the nonlinear terms $N(u)$ from X^α to X. The following lemma reduces this proof to the proof of the Lipschitz-continuity from $W^{k,q}$ to L^p, cf. [**Hen81**, Theorem 1.6.1].

Lemma 6.3.8. *Let $\Omega \subset \mathbb{R}^d$ be an open set with smooth boundary, let $1 \le p < \infty$, and let A be a sectorial operator in $X = L^p(\Omega)$ with $D(A) = X^1 \subset W^{m,p}(\Omega)$ for a $m \ge 1$. Then*

$$X^\alpha \subset W^{k,q} \qquad or \qquad X^\alpha \subset C^\nu(\Omega)$$

for $\alpha \in [0,1]$ if $k - d/q < m\alpha - d/p$, $q \ge p$ or $0 \le \nu < m\alpha - d/p$.

In order to prove that $D(A) = X^1 \subset W^{2,2}$, for $f \in L^2$ one has to find solutions $u \in W^{2,2}$ of the elliptic problem

$$-\Delta u + \nabla p = f, \qquad \nabla \cdot u = 0$$

in Ω with Dirichlet boundary conditions for u. The existence of such solutions is implied by elliptic regularity theory [**ADN59, ADN64**]. Since $D(A) \subset W^{2,2}$ it follows for $d = 3$ by Lemma 6.3.8 that for $\alpha \in (1/2,1)$ that $X^\alpha \subset W^{1,q}$ provided $1/q > (5-4\alpha)/6$ and that for $\alpha \in (3/4,1)$ that $X^\alpha \subset L^\infty$. Therefore

$$\|N(u)\|_X = \|N(u)\|_{L^2} \le \|u\|_{L^\infty}\|\nabla u\|_{L^2} \le C\|u\|_{X^\alpha}^2$$

for $\alpha \in (3/4,1)$. For $d = 2$ we find $\alpha \in (1/2,1)$. Hence, we have the Lipschitz-continuity of the polynomial $N(u)$ from X^α to X for $d = 3$ if $\alpha \in (3/4,1)$ and for $d = 2$ if $\alpha \in (1/2,1)$.

Theorem 6.3.9. *Let $\alpha \in (3/4,1)$ if $d = 3$ or $\alpha \in (1/2,1)$ if $d = 2$. For $u_0 \in X^\alpha$ there exists a $T_0 > 0$ such that the Navier-Stokes equations possess a unique mild solution $u \in C([0,T_0], X^\alpha)$ with $u|_{t=0} = u_0$.*

In the proof of the global existence of solutions $u \in C([0,T_0], H^{1,div}_{\text{per}})$ of the Navier-Stokes equations in $d = 2$ space dimensions in §6.2.3 we used (6.19) which only holds in $\mathbb{R}^2$ and periodic boundary conditions. In this section we prove the global existence without using (6.19). The method gives weaker estimates but it is more general and periodic boundary conditions are not needed; nevertheless we keep them for simplicity.

With the Gagliardo-Nirenberg inequality L^p-norms can be estimated by L^q-Norms and gradients, for $p > q$. Such inequalities are called interpolation estimates since a norm in the middle (L^p) is interpolated with the help of a weaker norm (L^q) and a stronger norm (gradient). Here we give a simple version. See, e.g., [**Hen81**, Page 37] for a general version, and Exercise 6.6 for the proof of an even weaker version.

Lemma 6.3.10. (A simple Gagliardo-Nirenberg inequality) *For $d = 1, 2, 3, 4$ we have*

$$\|u\|_{L^4(\mathbb{T}^d)} \leq C\|u\|_{L^2(\mathbb{T}^d)}^{1-d/4}\|u\|_{H^1(\mathbb{T}^d)}^{d/4}.$$

With the help of Lemma 6.3.10 we can proceed as follows. The L^2-estimate in domains with general Lipschitz-continuous boundary and Dirichlet boundary conditions works exactly the same as before. For the H^1-norm we estimate again as before

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}|\nabla u|^2\,\mathrm{d}x \leq -\int_{\mathbb{T}^d}|\Delta u|^2\,\mathrm{d}x + g(u) \tag{6.22}$$

with $g(u) = \left|\int_{\mathbb{T}^d}(\Delta u)\cdot((u\cdot\nabla)u)\,\mathrm{d}x\right|$. For periodic boundary conditions and $d = 2$ we have $g(u) = 0$. For general boundary conditions and/or $d = 3$ the best estimate known is

$$\begin{aligned} g(u) &\leq \|\Delta u\|_{L^2}\|(u\cdot\nabla)u\|_{L^2} \leq \|\Delta u\|_{L^2}\left(\int_{\mathbb{T}^d}|\nabla u|^2|u|^2\,\mathrm{d}x\right)^{1/2} \\ &\leq \|\Delta u\|_{L^2}\|u\|_{L^4}\|\nabla u\|_{L^4} \end{aligned}$$

where we used the Cauchy-Schwarz inequality. The Gagliardo-Nirenberg estimate and the Poincaré-inequality $\|u\|_{H^m} \leq C\sum_{|\alpha|=m}\|\partial_x^\alpha u\|_{L^2}$, see §5.2.2, give

$$\begin{aligned} g(u) \leq& \|\Delta u\|_{L^2}\left(C\|u\|_{L^2}^{1-d/4}\|u\|_{H^1}^{d/4}\right)\left(C\|\nabla u\|_{L^2}^{1-d/4}\|\nabla u\|_{H^1}^{d/4}\right) \\ \leq& C\|\Delta u\|_{L^2}^{1+d/4}\|u\|_{L^2}^{1-d/4}\|\nabla u\|_{L^2}. \end{aligned}$$

In order to balance the factor $\|\Delta u\|_{L^2}^2$ on the right-hand side against $-\|\Delta u\|_{L^2}^2$ in (6.22) we use Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for } a, b \geq 0 \text{ and } p, q > 1 \text{ with } 1/p + 1/q = 1. \tag{6.23}$$

Case $d = 2$: We choose $\varepsilon = (4/3)^{3/4}$ and obtain

$$g(u) \leq \frac{C}{\varepsilon}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\ \varepsilon\|\Delta u\|_{L^2}^{3/2}$$

$$\leq \frac{1}{4}\left(\frac{C}{\varepsilon}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\right)^4 + \frac{3\varepsilon^{4/3}}{4}\|\Delta u\|_{L^2}^2$$
$$\leq \|\Delta u\|_{L^2}^2 + C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4$$

using (6.23) with $q = 4/3$ and $p = 4$. Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 \leq a(t)\|\nabla u\|_{L^2}^2, \qquad \text{where} \qquad a(t) = 2C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2.$$

From

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 = -\|\nabla u\|_{L^2}^2,$$

cf. §6.2.3, follows $\|u(t)\|_{L^2} = \mathrm{e}^{-t}\|u_0\|_{L^2}$, and then

$$\int_0^t \|\nabla u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau = \|u_0\|_{L^2}^2 - \|u(t)\|_{L^2}^2.$$

Note, that this estimate does not imply a uniform bound for $h(t) = \|\nabla u(t)\|_{L^2}$, but its square-integrability. As a consequence,

$$0 \leq \int_0^t a(t) \leq C \sup_{\tau\in(0,t)} \|u(\tau)\|_{L^2}^2 \int_0^t \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau < \infty$$

uniformly for all $t \geq 0$ and so

$$\|\nabla u(t)\|_{L^2}^2 \leq \mathrm{e}^{\int_0^t a(\tau)\,\mathrm{d}\tau}\|\nabla u_0\|_{L^2}^2 \leq M\|\nabla u_0\|_{L^2}^2$$

for a $M \geq 0$ independent of $t \geq 0$. Therefore, the local solution can be extended to a global solution, i.e., we obtain the global existence in 2D.

Case $d = 3$: Similar to the 2D case we use the Gagliardo-Nirenberg estimate and Young's inequality with $q = 8/7$ and $p = 8$ and obtain

$$g(u) = \left|\int_{\mathbb{T}^d} (\Delta u)\cdot((u\cdot\nabla)u)\,\mathrm{d}x\right| \leq \|\Delta u\|_{L^2}\|u\|_{L^4}\|\nabla u\|_{L^4}$$
$$\leq C\|\Delta u\|_{L^2}^{7/4}\|\nabla u\|_{L^2}\|u\|_{L^2}^{1/4} \leq C\left(\|\nabla u\|_{L^2}\|u\|_{L^2}^{1/4}\right)^8 + \|\Delta u\|_{L^2}^2.$$

If we proceed as above it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 \leq a(t)\|\nabla u\|_{L^2}^2, \qquad \text{where} \qquad a(t) = 2C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^6.$$

The equation for the dissipation only guarantees that the function $h(t) = \|\nabla u(t)\|_{L^2}$ is in $L^2((0,T_0))$, but not in $L^6((0,T_0))$, and therefore we cannot proceed as above. For a further discussion we refer to [**Wie99, Con01**].

Remark 6.3.11. Since so far it cannot be proved that for $d = 3$ unique global solutions exist, it appears to be nonsense to discuss their long time dynamics. Nevertheless, if one assumes the existence of global strong solutions, then the concept of attractor is again quite useful. In particular, there are a number of estimates for the dimension of attractors (which is finite) for the Navier-Stokes equations in two and three space dimensions, cf. [**Tem97**]. However, a finite-dimensional attractor by no means implies that the dynamics is "simple". In fact there is a lot of theory on turbulent flows and also on so called fully developed turbulence which is mainly based on methods from statistical physics, cf. [**FRMT01**]. ⌋

Further Reading. Classical books about local existence and uniqueness of the solutions of the Navier-Stokes equations are [**vW85, Tem01**]. More background on the derivation and applications of the Navier-Stokes equations and related equations can be found in [**Fow97**]. Concise treatments of these equations as a dynamical system, embedded in the general existence theory of semilinear parabolic equations can be found for instance in [**Hen81, DG95, Tem97, Rob01**]; the latter three are also recommended for the so called Galerkin method as an alternative to semigroup methods for proving local existence in the Navier-Stokes equations and general parabolic systems.

An excellent textbook, going way beyond the brief summary given here, is [**MB02**]. An essential reference for steady problems in exterior domains, including the necessary function spaces and inequalities, is [**Gal11**]. The dynamics and stability of vortices is treated in [**MB02, GW05, GW06b**]. See [**WW15**, Chapter 1] for a very accessible account on metastable states and the finite dimensionality of the global attractor for the 2D Navier-Stokes equations. An enlightening essay about the Navier-Stokes Millenium problem is [**Tao09**], emphazising the scale invariance, see Exercise 6.7. Finally, [**Lem16**] gives an impressive overview about the state of the art of the mathematical analysis of the Navier-Stokes equations, and an excellent review of the Millennium problems, with focus on the Navier-Stokes equations.

Exercises

6.1. (a) Show that $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $\nabla \times u = 0$ and $\nabla \cdot u = 0$ is equivalent to the Cauchy-Riemann differential equations for $w(z) = u_1(x, y) - \mathrm{i}u_2(x, y)$, $z = x + \mathrm{i}y$. (b) Sketch the flow belonging to $w(z) = z^2$ and calculate the associated pressure.

6.2. (d'Alembert Paradox) Let $a > 0$, $U \in \mathbb{R}^3$, and $\phi(x) = \left(\frac{a^3}{2\|x\|^3} + 1\right) \langle U, x\rangle$ for $x \in \Omega = \mathbb{R}^3 \setminus B_a(0)$. Sketch $u = \nabla\phi$, show that $\operatorname{div} u = 0$, and calculate the drag $f = -\int_{B_a(0)} \phi n \,\mathrm{d}S$.

6.3. For given $\omega = \omega(x)$, $x \in \mathbb{T}^2$, find in Fourier space an explicit solution u of

$$\nabla \times u = \omega \quad \text{and} \quad \nabla \cdot u = 0.$$

6.4. Show that $\|\widehat{u} * \widehat{v}\|_{\ell^\omega_{1,\sqrt{t}}} \leq \|\widehat{u}\|_{\ell^\omega_{1,\sqrt{t}}} \|\widehat{v}\|_{\ell^\omega_{1,\sqrt{t}}}$.

6.5. Show the enstrophy identity (6.19).

6.6. Prove the following weak form of the Gagliardo-Nirenberg inequality for $(2\pi)^d$-periodic functions. For $d = 1, 2, 3$ and $\delta > 0$ we have

$$\|u\|_{L^4(\mathbb{T}^d)} \leq C\|u\|_{L^2(\mathbb{T}^d)}^{1-d/4-\delta}\|u\|_{H^1(\mathbb{T}^d)}^{d/4+\delta}.$$

Hint: Use $\|u\|_{L^p} \leq C\|\widehat{u}\|_{L^q}$ for $1/p+1/q = 1$ and $q \in [1, 2]$ and the Hölder inequality

$$\sum_{k\in\mathbb{Z}} |a_k b_k c_k| \leq (\sum_{k\in\mathbb{Z}} |a_k|^{p_1})^{1/p_1}(\sum_{k\in\mathbb{Z}} |b_k|^{p_2})^{1/p_2}(\sum_{k\in\mathbb{Z}} |c_k|^{p_3})^{1/p_3}$$

with $1/p_1 + 1/p_2 + 1/p_3 = 1$.

6.7. Let $(u, p) : \mathbb{R}^d \times [0, T) \to \mathbb{R}^d \times \mathbb{R}$ be a solution of the d-dimensional Navier-Stokes equations over $\mathbb{R}^d$. For $l > 0$ set $u^l(x, t) = l^{\alpha_1}u(l^\beta x, l^\gamma t)$ and $p^l(x, t) = l^{\alpha_2}p(l^\beta x, l^\gamma t)$. Find $\alpha_{1,2}, \beta, \gamma$, and δ such that (u^l, p^l) is again a solution of the Navier-Stokes equation. Discuss how the energy and energy dissipation behave under this scaling.

Chapter 7

Some dissipative PDE models

This is the first chapter of Part III of this book. Here and in the remainder of this book we consider PDEs on unbounded domains. In order to avoid dealing with far away boundaries, whose influence on the solutions in the interior of the domain is small at least for a long time, we idealize the large domain to an unbounded domain. For instance, instead of x from some large interval $(-L, L)$ we consider $x \in \mathbb{R}$. From a didactic point of view the consideration of unbounded domains has certain advantages. Since we do not have to deal with boundary conditions which are often a source of functional analytic difficulties, this idealization allows to explain genuine PDE phenomena such as transport, diffusion and dispersion. Hence, it allows us to keep the functional analytic tools at a minimum. Unbounded domains are easy in this respect.

On the other hand, compared to PDEs over bounded domains there are new fundamental and challenging open questions, mainly due to the fact that PDEs on unbounded domains define dynamical systems with uncountably many modes (degrees of freedom). In contrast to the situation of countably many modes considered in Chapters 5 and 6, a separation of the uncountably many modes into single modes is a highly singular action from a functional analytic point of view, and therefore in general of little use. The recovery of compactness by smoothing properties is no longer true, and therefore finite-dimensional attractors in general cannot be expected.

To illustrate our point of view, the following example shows that also for PDEs defined on a very large domain in space the interpretation as countably many ODEs is no longer a big help.

Example. Consider the linear wave equation $\partial_t^2 u = \partial_x^2 u$ for $t \in \mathbb{R}$, $x \in (-L, L)$, $L > 0$ very large, $u(x,t) \in \mathbb{R}$, with Dirichlet boundary conditions $u(-L,t) = u(L,t) = 0$. We consider two special classes of solutions, first the oscillations of the eigenfunctions,

$$u(x,t) = \sin(n\pi t/(2L)) \sin(n\pi(x-L)/(2L)), \quad n \in \mathbb{N},$$

and secondly the traveling wave solutions

$$u(x,t) = f(x-t) + g(x+t)$$

with f and g arbitrary smooth functions with compact support

$$\operatorname{supp}(f) = \{x : f(x) \neq 0\} \subset [-1,1] \qquad \text{and} \qquad \operatorname{supp}(g) \subset [-1,1].$$

As long as $|t| < L - 1$ this is a solution of the PDE, i.e., for a very large time interval traveling wave solutions play a role. An expansion of these solutions in eigenfunctions is of no use. $\rfloor$

In this Chapter 7 we start with some scalar model problems. These are the Kolmogorov-Petrovsky-Piskounov (KPP) or Fisher equation in §7.1, the Allen-Cahn equation in §7.2, and the Burgers equation in §7.4. Moreover, there is the method oriented §7.3 about Fourier transform. We keep the exposition rather brief and aim for a basic understanding of the various models. We are interested in the local existence and uniqueness of solutions and in the existence and stability of special solutions, which are important for the underlying physical processes which are described by the models. To construct these special solutions we often use the ODE methods from Part I. In order to make this part more self-contained we recall a number of definitions and constructions which are only small adaptions of respective concepts from Part I. In Chapter 8 we consider with the NLS, KdV, and the GL equation the three canonical modulation equations whose dynamics we will recover in more complicated PDEs in Part IV of this book. Part III of this book is closed with Chapter 9 about reaction-diffusion systems.

For each of the equations considered in this Part there already exists much literature on various levels, see the "further reading" at the end of each chapter.

7.1. The KPP equation

The Kolmogorov-Petrovsky-Piskounov (KPP) equation [**KPP37**] or Fisher equation [**Fis37**]

$$\partial_t u = \partial_x^2 u + u - u^2, \tag{7.1}$$

with $t \geq 0$, $x \in \mathbb{R}$, and $u = u(x,t) \in \mathbb{R}$, occurs as a model for various systems in nature, for instance for chemical reactions or population dynamics. The equation consists of two parts, namely the diffusion term $\partial_x^2 u$ and the nonlinear reaction term $u-u^2$. Therefore, it brings together PDE with ODE dynamics.

Inserting $u(x,t) = v(t)$ into (7.1) gives the one-dimensional ODE

$$\dot{v} = v - v^2. \tag{7.2}$$

The 1D phase portrait shows that the fixed point $v = 0$ is unstable and that the fixed point $v = 1$ is asymptotically stable. The term $+v$ in the KPP equation describes exponential growth for small v and the term $-v^2$ represents saturation. For instance, a population of animals or a chemical reaction initially increases with some exponential rate until the growth is saturated by the available food or the missing reactant. If $v(0) > 0$, then $\lim_{t\to\infty} v(t) = 1$.

Before we combine the ODE dynamics coming from the reaction term $u-u^2$ with the dynamics coming from the diffusion term $\partial_x^2 u$ we discuss the modeling and the properties of linear diffusion in the next two subsections.

7.1.1. The modeling of diffusion.

Diffusion occurs in various situations. We explain three such situations, namely Brownian motion, a discrete random walk, and Fourier's law.

Brownian motion and diffusion. The term Brownian motion is named after Robert Brown who in 1827 described the irregular motion of pollen particles suspended in water. In [**Ein05**] Einstein studied Brownian motion the following way. Consider a long, thin tube filled with clear water, into which we inject at time $t = 0$ a unit amount of ink, at the location $x = 0$. Let $u(x,t)$ denote the density of ink at position $x \in \mathbb{R}$ and time $t \geq 0$. Suppose that the probability that an ink particle moves from x to $x+y$ in a time τ is translational invariant, i.e., does not depend on x. This probability is denoted by $\rho(y,\tau)$. Then

$$\begin{aligned} u(x,t+\tau) &= \int_{\mathbb{R}} u(x-y,t)\rho(y,\tau)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \left(u(x,t) - (\partial_x u(x,t))y + \frac{1}{2}(\partial_x^2 u(x,t))y^2 + \dots \right) \rho(y,\tau)\,\mathrm{d}y. \end{aligned}$$

Now $\rho(-y,\tau) = \rho(y,\tau)$ by symmetry such that $\int_{\mathbb{R}} y\rho(y,\tau)\,\mathrm{d}y = 0$. Next assume that the variance is linear in τ, i.e.,

$$\int_{\mathbb{R}} y^2 \rho(y,\tau)\,\mathrm{d}y = 2D\tau \tag{7.3}$$

for a $D > 0$. Then

$$\frac{1}{\tau}(u(x,t+\tau) - u(x,t)) = D\partial_x^2 u(x,t) + \text{h.o.t.}$$

Under the assumption that all higher moments of ρ decay faster than τ for $\tau \to 0$, in the limit $\tau \to 0$ we obtain the linear diffusion equation

$$\partial_t u = D\partial_x^2 u. \tag{7.4}$$

Einstein derived the relation $2D = RT/(N_A\nu)$ where R is the gas constant, T the temperature, N_A the Avogadro number, and ν a friction coefficient.

A discrete random walk. We consider a two-dimensional rectangular lattice, comprising the sites $\{(m\delta x, n\delta t) : m = 0, \pm1, \pm2, \ldots; n = 0, 1, 2, ...\}$. A particle starting in $x = 0$ at a time $t = 0$ decides at each time $n\delta t$ to move an amount δx to the left or to move an amount δx to the right, both possibilities with probability $1/2$. Denote by $p(m, n)$ the probability that the particle is at the position $m\delta x$ at the time $n\delta t$. Then $p(0, 0) = 1$ and $p(m, 0) = 0$ for $m \neq 0$. Also, $p(m, n + 1) = \frac{1}{2}(p(m - 1, n) + p(m + 1, n))$, hence

$$p(m, n + 1) - p(m, n) = \frac{1}{2}(p(m - 1, n) - 2p(m, n) + p(m + 1, n)).$$

Now assume that $(\delta x)^2/(\delta t) = 2D$ which corresponds to (7.3) above. Then

$$\frac{1}{\delta t}(p(m, n + 1) - p(m, n)) = \frac{D}{(\delta x)^2}(p(m - 1, n) - 2p(m, n) + p(m + 1, n)),$$

and sending $\delta \to 0$ again yields the linear diffusion equation (7.4).

Fourier's law. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be the temperature inside a material body and let $V \subset \mathbb{R}^3$ be a test volume with surface S. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V u\,\mathrm{d}V = -\int\limits_S j \cdot n\,\mathrm{d}S = -\int\limits_V \text{ div } j\,\mathrm{d}V\ ,$$

where $j = j(x, t) \in \mathbb{R}^3$ is the heat flow. Since this is true for all test volumes V we find

$$\partial_t u + \text{ div } j = 0\ .$$

It is reasonable to assume that the heat flow from warm to cold is proportional to the negative temperature gradient, i.e., $j = -D\nabla u$. This so called Fourier's law again yields the heat equation

$$\partial_t u = D \text{ div } \nabla u = D\Delta u.$$

7.1.2. Diffusion on the real line. Throughout this subsection we consider the linear diffusion equation with diffusion coefficient $D = 1$, i.e.,

$$\partial_t u = \partial_x^2 u, \tag{7.5}$$

which always can be achieved by a rescaling of time or space. We already observed the dissipative character of this equation in previous sections where we found the solutions $u(x, t) = \mathrm{e}^{-k^2 t}\sin(kx)$ which decay to zero for $|k| > 0$ with some exponential rate.

For $t > 0$ the general formula for the solutions of the linear diffusion equation (7.5) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4t}} u(y,0)\,\mathrm{d}y. \tag{7.6}$$

The derivation of this formula is given subsequently, but also by a different method in §7.3. The existence of this integral is guaranteed for $t > 0$ if for instance $\sup_{y\in\mathbb{R}} |u(y,0)| < \infty$.

From (7.6) we immediately obtain the estimate

$$\sup_{x\in\mathbb{R}} |u(x,t)| \le \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |u(x,0)|\,\mathrm{d}x,$$

i.e., solutions to spatially localized initial conditions decay uniformly towards zero with a rate $t^{-1/2}$. Since mass is conserved, i.e.,

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = \int_{-\infty}^{\infty} u(x,0)\,\mathrm{d}x$$

for all $t \ge 0$, this is how diffusion is expected to work. The conservation of mass follows for instance with the use of the solution formula from

$$\begin{aligned}\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}x \right) u(y,0)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} 1 \cdot u(y,0)\,\mathrm{d}y.\end{aligned}$$

The decay happens in a universal manner. The initial condition

$$u(y,0) = \delta_0(y),$$

with δ_0 the "δ-distribution in $x = 0$", cf. Example 5.2.2, leads to the self-similar solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}}. \tag{7.7}$$

The solution only exists for $t > 0$. This is a general rule. For an arbitrary initial condition the diffusion equation cannot be solved backwards in time. Since this solution is the starting point of the construction of the general solution formula it is also called fundamental solution. In order to derive (7.7) we make the ansatz $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$ and find $v'' = -\frac{1}{2}\xi v'$ which is solved by $v' = c\mathrm{e}^{-\frac{\xi^2}{4}}$ with a constant $c \in \mathbb{R}$. Since with $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$ also $\partial_x u = \frac{1}{\sqrt{t}} v'\left(\frac{x}{\sqrt{t}}\right)$ is a solution of (7.5) we find (7.7).

In lowest order self-similar behavior is also observed for general spatially localized initial conditions, namely,

$$u(x,t) = \frac{A^*}{\sqrt{t}} v\left(\frac{x}{\sqrt{t}}\right) + \mathcal{O}(1/t) \quad \text{with} \quad v(\xi) = \mathrm{e}^{-\xi^2/4} \tag{7.8}$$

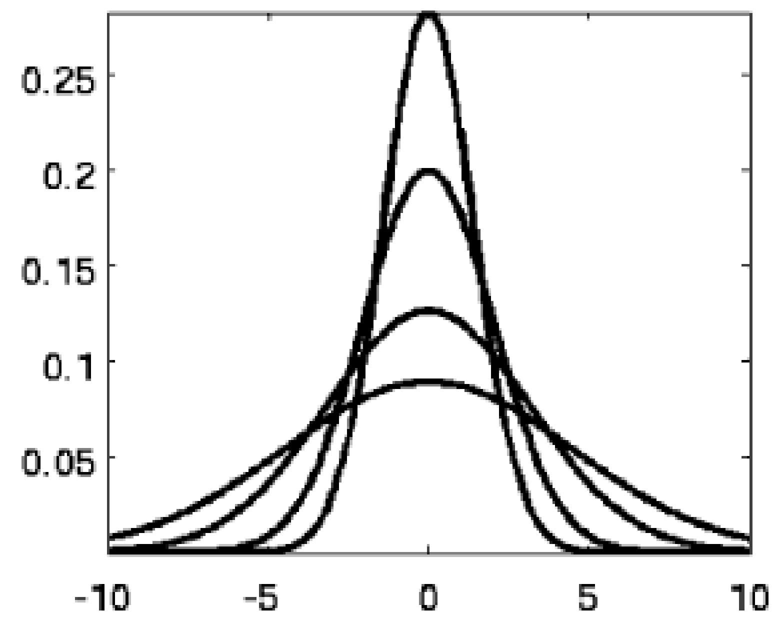


Figure 7.1. Self-similar decay of the special solution $u(x,t) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{x^2}{4t}}$ for $t \in \{1,2,5,10\}$.

and with a constant $A^* \in \mathbb{R}$ depending on the initial conditions. This is explained in detail in §14.

Diffusion is smoothing. We obtain the estimate

$$\begin{aligned}\sup_{x\in\mathbb{R}} |\partial_x u(x,t)| &= \sup_{x\in\mathbb{R}} \Big| \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} -\frac{(x-y)}{2t} \mathrm{e}^{-\frac{(x-y)^2}{4t}} u(y,0)\, \mathrm{d}y \Big| \\ &= \sup_{x\in\mathbb{R}} \Big| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} -\frac{s}{\sqrt{t}} \mathrm{e}^{-s^2} u\left(x - 2\sqrt{t}s, 0\right) \mathrm{d}s \Big| \\ &\leq \frac{1}{\sqrt{t}} \frac{2}{\sqrt{\pi}} \int_0^{\infty} s\mathrm{e}^{-s^2}\, \mathrm{d}s \ \sup_{x\in\mathbb{R}} |u(x,0)| \leq \frac{C}{\sqrt{t}} \sup_{x\in\mathbb{R}} |u(x,0)|,\end{aligned}$$

with a constant C independent of t and of $u(\cdot,0)$, where we made the transformation $s = (x-y)/(2\sqrt{t})$, i.e., $\mathrm{d}s = -\,\mathrm{d}y/(2\sqrt{t})$. This can easily be generalized.

Theorem 7.1.1. *Let $u = u(\cdot,t)$ be a solution of the linear diffusion equation. Then for all $n \in \mathbb{N}$ there exists a $C > 0$ such that for all $t > 0$*

$$\|\partial_x^n u(\cdot,t)\|_{C_b^0} \leq C t^{-n/2} \|u(\cdot,0)\|_{C_b^0}.$$

Finally for every $t_0 > 0$ and $x_0 \in \mathbb{R}$ the function $(x,t) \mapsto u(x,t)$ can be expanded in a convergent power series around (x_0,t_0), i.e., u is an analytic function and can be extended into the complex plane. See §6.2.2.

In order to handle the linear diffusion equation $\partial_t u = \partial_x^2 u$ with dynamical systems methods we have to choose a suitable phase space. We already know that in infinite dimensions the choice of phase space and associated norm is fundamental. A first choice is $X = C^0_{b,unif}(\mathbb{R},\mathbb{R})$, the space of uniformly bounded and uniformly continuous functions $u : \mathbb{R} \to \mathbb{R}$, equipped with the norm

$$\|u(t)\|_{C^0_{b,unif}} = \sup_{x\in\mathbb{R}} |u(x,t)|\,.$$

Lemma 7.1.2. *The curve $t \mapsto u(t, u_0)$ is continuous in X if $u_0 \in X$.*

Proof. Since $u(t+s, u_0) = u(t, u(s, u_0))$ it is sufficient to prove the continuity of the orbit $t \mapsto u(t)$ for $t \searrow 0$ in the space X. With $H(z) = \frac{1}{\sqrt{4\pi}} \mathrm{e}^{-z^2/4}$ we estimate

$$\begin{aligned} \|u(t, u_0) - u_0\|_{C_b^0} &= \sup_{x \in \mathbb{R}} |u(x,t) - u(x,0)| \\ &= \sup_{x \in \mathbb{R}} \Big| \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} H\left(\frac{x-y}{\sqrt{t}}\right) (u(y,0) - u(x,0)) \, \mathrm{d}y \Big| \\ &= \sup_{x \in \mathbb{R}} \Big| \int_{-\infty}^{\infty} H(z) \Big(u(x - \sqrt{t}z, 0) - u(x,0) \Big) \, \mathrm{d}z \Big| \\ &\leq \sup_{x \in \mathbb{R}} \int_{|z| \leq R} |\dots| \, \mathrm{d}z + \sup_{x \in \mathbb{R}} \int_{|z| \geq R} |\dots| \, \mathrm{d}z = s_1 + s_2. \end{aligned}$$

For a given $\varepsilon > 0$ we have to find a $t_0 > 0$ such that for all $t \in (0, t_0)$ we have $s_1 + s_2 < \varepsilon$. We can estimate

$$s_2 \leq 2 \int_{|z| \geq R} H(z) \, \mathrm{d}z \ \sup_{x \in \mathbb{R}} |u(x,0)| < \varepsilon/2$$

by choosing an $R > 0$ sufficiently large due to the definition of H. Next, we estimate

$$\begin{aligned} s_1 &\leq \int_{-\infty}^{\infty} H(z) \, \mathrm{d}z \sup_{x \in \mathbb{R}, |z| \leq R} |u(x + \sqrt{t}z, 0) - u(x,0)| \\ &= \sup_{x \in \mathbb{R}, |z| \leq R} |u(x + \sqrt{t}z, 0) - u(x,0)|. \end{aligned}$$

Since $x \mapsto u(x,0)$ is uniformly continuous for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in \mathbb{R}$ with $|y| < \delta$ we have $|u(x+y,0) - u(x,0)| < \varepsilon/2$. Choosing $t_0 > 0$ so small that $t_0 R < \delta$ we are done. $\square$

The deeper reason why $X = C_b^0$ would not be a good choice is explained in the following remark, but for a slightly simpler PDE.

Remark 7.1.3. If we consider the translation semigroup $T(t) : u(\cdot) \mapsto u(\cdot + t)$ which is the solution operator of the transport equation $\partial_t u = \partial_x u$ in the space $X = C_b^0(\mathbb{R}, \mathbb{R})$, we have that $T(t)$ is not a C_0-semigroup, cf. Definition 5.1.9, since for $u(x) = \sin(x^2)$ which is an element of C_b^0, but not of $C_{b,unif}^0$ we always have $\|u(\cdot) - u(\cdot + t)\|_{C_b^0} = 2$, if $t > 0$. The same is true for the linear diffusion equation. The deeper reason for the non-continuity of both semigroups in C_b^0 is the fact that the domain of definition C_b^1, respectively C_b^2, is not dense in C_b^0, cf. the theorem of Hille-Yosida, cf. [**Paz83**, Section 1.3, Theorem 3.1]. In order to have a C_0-semigroup we have to restrict to $X = C_{b,unif}^0(\mathbb{R}, \mathbb{R})$, which excludes the counter-example

$u(x) = \sin(x^2)$, where the faster and faster oscillations for $|x| \to \infty$ destroy the uniform continuity w.r.t. x and t. ⌋

Remark 7.1.4. With $(x,t) \mapsto u(x,t)$ a solution of (7.5) also $(x,t) \mapsto u(x+y,t)$ is a solution of (7.5), i.e., every solution shifted by y is a solution, too. More abstractly, the solution operator and the translation operator commute. With $t \mapsto u(x,t)$ a solution, obviously every derivative $t \mapsto \partial_x^n u(x,t)$ and every integral is a solution, too.

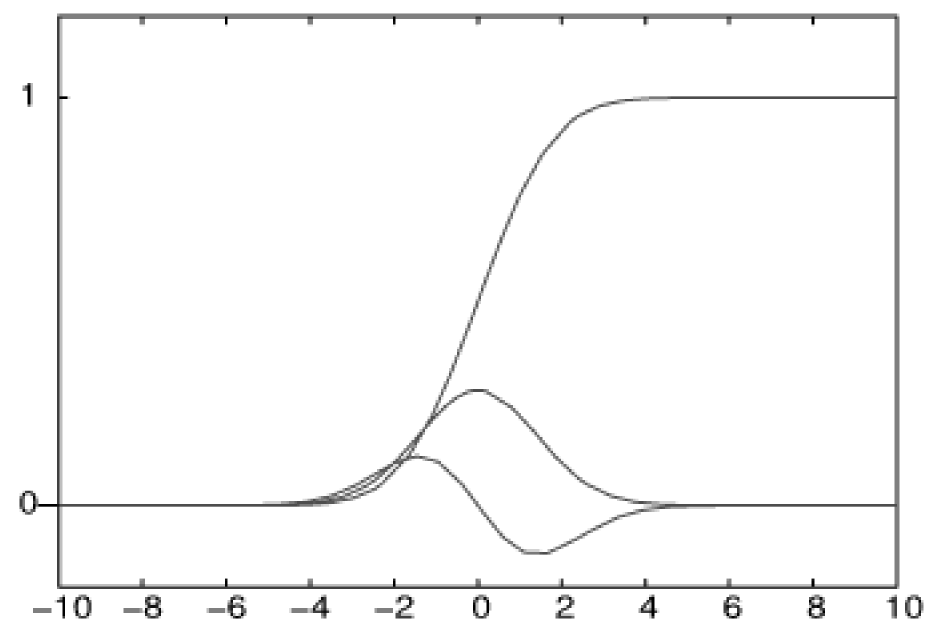


Figure 7.2. The solutions $u(x,t) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}}$, $\partial_x u$ and its first integral for $t = 1$.

Moreover, every linear combination and every convergent series of solutions, or every convergent integral over a set of solutions are again solutions. These properties are used in the following paragraph. ⌋

Our starting point to derive the solution formula (7.6) is the explicit solution (7.7). With this solution also

$$u(x,t) = \frac{1}{\sqrt{t}} \mathrm{e}^{-\frac{(x-y)^2}{4t}} \qquad \text{and} \qquad u(x,t) = \int_{-\infty}^{\infty} \frac{c(y)}{\sqrt{t}} \mathrm{e}^{-\frac{(x-y)^2}{4t}} \, dy$$

are solutions of (7.5). From the limit $t \to 0$ and Lemma 7.1.2 we find that $c(y) = \frac{u(y,0)}{\sqrt{4\pi}}$. The solution formula can be interpreted as linear combination of fundamental solutions, i.e., of diffusion processes starting in every point $x \in \mathbb{R}$ with a δ-distribution as initial condition.

7.1.3. The reaction-diffusion equation. The KPP equation is obtained by adding the diffusion term to the ODE (7.2). Thus, the KPP equation describes for instance a chemical reaction or the evolution of a disease in a large, here in an infinitely extended, domain, where the concentration $u = u(x,t)$ spreads into space by diffusion.

In order to handle the KPP equation as an abstract dynamical system we choose the same phase space as for the linear diffusion equation, namely $X = C^0_{b,\mathrm{unif}}(\mathbb{R}, \mathbb{R})$. Solutions $u = u(t) \in X$ satisfy the KPP equation only in a weak sense.

ISBN 978-1-4704-3613-1
9 781470 436131