Introduction to Stochastic Partial Differential Equations

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Historical Remarks

1. Stochastic Differential Equations

In 1940's, K.Itô introduced the now well-known Itô equation in $\mathbf{R}^d$:

$$\begin{aligned} dx(t) &= b(x(t),t)\,dt + \sigma(x(t),t)\,dw_t, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where w_t is a n-dimensional Brownian motion, $b : \mathbf{R}^d \times [0,T] \to \mathbf{R}^d$, and $\sigma : \mathbf{R}^d \times [0,T] \to \mathbf{R}^{d\times m}$ This equation is equivalent to the following integral equation:

$$x(t) = \xi + \int_0^t b(x(s),s)\,ds + \int_0^t \sigma(x(s),s)d\,w_s, \quad 0 \le t \le T. \tag{2}$$

2. Statistical Theory of Turbulence

From 1940 to 1960's, theory of turbulence was a very active research area in fluid dynamics.

A mathematical model was first proposed in 1952 by E. Hopf as a stochastic initial-value problem for the the following Navier-Stokes equation with $\sigma = 0$:

$$
\begin{gathered}
\frac{\partial}{\partial t} u(x,t) + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \triangle u + \sigma \dot{W}(x,t), \\
\nabla \cdot u = 0, \quad x \in D, \ t > 0, \\
u|_{\partial D} = 0, \quad u(x,0) = g(x,\omega),
\end{gathered} \tag{3}
$$

where $W(x,t)$ is a R-Wiener random field in $\mathbf{R}^3$. He derived the so-called Hopf equation for the characteristic function : $\Phi(t,\lambda) = \mathbb{E}\{exp\{i(u_t,\lambda)\}$. For $\sigma \neq 0$, the generalized Hopf equation takes the following form:

$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi(t,\lambda) = \sum_{j,k=1}^{3}\int_D\int_D \frac{\partial \lambda_j(x)}{\partial x_k}\frac{\delta^2\Phi(t,\lambda)}{\delta\lambda_j(x)\delta\lambda_k(y)}dxdy \\
-\nu\sum_{j,k=1}^{3}\frac{\partial \lambda_j(x)}{\partial x_k}\frac{\partial}{\partial x_k}\frac{\delta\Phi(t,\lambda)}{\delta\lambda_j(x)}dx \\
-\frac{1}{2}\sum_{j,k=1}^{3}\int_D\int_D r_{j,k}(x,y)\lambda_j(x)\lambda_k(y)\,dxdy\,\Phi(t,\lambda),
\end{aligned}
\tag{4}
$$

$$
\Phi(0,\lambda) = \Phi_0(\lambda).
$$

Existence and uniqueness results proved by C. Foias (1974), M.Visik and V.I. Komech (1981)...

3.Turbulence Related Problems

Turbulent Diffusion

$$\frac{\partial u}{\partial t} = \nu \Delta u - \sum_{k=1}^{3} v_k(t,x,\omega)\frac{\partial u}{\partial x_k} + q(x,t),$$
$$\frac{\partial u}{\partial n}|_{\partial D} = 0, \quad u(x,0) = u_0(x), \tag{5}$$

where $v(t,x,\omega)$ is the turbulent velocity.

Stochastic Wave Propagation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + \sum_{k=1}^{3} v_k(t,x,\omega)\frac{\partial u}{\partial x_k} + q(x,t),$$
$$\frac{\partial u}{\partial n}|_{\partial D} = 0, \quad u(x,0) = u_0(x), \ \frac{\partial}{\partial t} u(x,0) = u_1(x), \tag{6}$$

where $v(t,x,\omega)$ is the turbulent velocity.

Stochastic PDEs of Itô Type

In particular, let $v(t,x,\omega) = \frac{\partial}{\partial t} W(t,x)$, where $W(t,x)$ is a Wiener random field in $\mathbb{R}^3$ Then the turbulent diffusion equation yields Itô's evolution equations or the stochastic evolution Equation:

$$\frac{\partial u}{\partial t} = \nu \Delta u + q(x,t) - \sum_{k=1}^{3} \left(\frac{\partial u}{\partial x_k}\right) \frac{\partial}{\partial t} W_k(t,x),$$
$$\frac{\partial u}{\partial n}\Big|_{\partial D} = 0, \quad u(x,0) = u_0(x),$$

or

$$u(x,t) = u_0(x) + \int_0^t \{\nu \Delta u(x,s) + q(x,s)\}\, ds - \sum_{k=1}^{3} \int_0^t \left(\frac{\partial u(x,s)}{\partial x_k}\right) W_k(ds,x)$$

Stochastic Evolution Equations in Hilbert Spaces

In 1972-3, A. Bensussan and R.Temam treated the Navier-Stokes equations with additive noise as a stochastic evolution equation in Hilbert space ($\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}^*$) in the form:

$$\frac{du_t}{dt} = Au_t + B(u_t) + f(t,\omega), \quad u_0 = \eta(\omega).$$

where $A : \mathscr{V} \to \mathscr{V}^*, B : \mathscr{V} \to \mathscr{V}^*$, and $f(t,\omega) = \dot{W}(t,x)$.

(**Example):**

$$\frac{\partial u}{\partial t} = \kappa \Delta u + g(u) + \frac{\partial}{\partial t} W(x,t),$$

$$u|_{\partial D} = 0, \quad u(x,0) = \eta(x,\omega), \quad x \in D, t \in (0,T),$$

where $A = \kappa\Delta, B(u) = g(u)$ **,**
$\mathscr{V} = H_0^1(D), \mathscr{H} = L^2(D), \mathscr{V}^* = H^{-1}(D)$.

Multiplicative Noise and Itô's Formula

$$du_t = Au_t dt + B(u_t)dt + \Sigma(u_t)dW_t, \quad u_0 = \eta(\omega).$$

Under some conditions, such as $B(u)$ is monotone,
E. Pardoux (1975) proved the existence and uniqueness of a **strong solution**:

$$u \in L^p(\Omega \times (0,T), \mathscr{V}) \cap L^p(\Omega, C([0,T], \mathscr{H})).$$

More importantly, he proved the Itô's formula:

$$\begin{aligned}\Phi(u_t,t) = \Phi(u_0,0) + \int_0^t \partial_s \Phi(u_s,s)\,ds + \int_0^t \langle Au_s, \Phi'(u_s,s)\rangle ds \\ + \int_0^t (B(u_s), \Phi'(u_s,s))ds + \int_0^t (\Phi'(u_s,s), dW_s) \\ + \frac{1}{2}\int_0^t Tr[R\Sigma^*(u_s)\Phi''(u_s,s)\Sigma(u_s)]ds.\end{aligned}$$

Mild Solutions

In 1980's, G. Da Prato introduced the semi-group approach:

$$du_t = Au_t dt + B(u_t)dt + \Sigma(u_t)dW_t, \quad u_0 = \eta(\omega),$$

where A generates a strongly continuous semigroup $\{G_t = e^{tA}, t \geq 0\}$ on $\mathscr{H}$. Rewrite the above equation as the stochastic integral equation:

$$u_t = G_t u_0 + \int_0^t G_{t-s} B(u_s)ds + \int_0^t G_{t-s} \Sigma(u_s)dW_s.$$

Its solution $u \in L^p(\Omega, C([0,T]), \mathscr{H})$ is a mild solution.

Contributions to the modern theory of Stochastic PDEs:

• Organizer of Trento International Conference Series (1985 – 2000) on Stochastic PDEs and their Applications.

• Book (1995, Da Prato and Zabczyk): Stochastic Equations in Infinite Dimensions.

Mathematical Questions

Stochastic Heat Equation

$$\frac{\partial u}{\partial t} = (\kappa\Delta - \alpha)u + \dot{W}(x,t), \quad x \in D, t \in (0,T),$$

$$u|_{\partial D} = 0, \quad u(x,0) = h(x).$$

Let $\{\phi_k, \lambda_k\}$ be the O-N set of eigen-pairs of $(\kappa\Delta - \alpha)$ with $\phi_k|_{\partial D} = 0$. Assume that $W(x,t) = \sum_{k=1}^{\infty} \sigma_k \phi_k(x) w_t^k$ with $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, where $\{w_t^k\}$ are iid Brownian motions in one dimension. We can obtain a formal solution:

$$u(x,t) = \sum_{k=1}^{\infty} u_t^k \phi_k(x),$$

where $u_t^k = h_k e^{-\lambda_k t} + \sigma_k \int_0^t e^{-\lambda_k (t-s)} dw_s^k, \quad k = 1, 2, ...$

The formal solution is given by

$$
\begin{aligned}
& u(x,t) = \hat{u}(x,t) + v(x,t) \\
= \quad & \sum_{k=1}^{\infty} h_k e^{-\lambda_k t} \phi_k(x) + \sum_{k=1}^{\infty} \sigma_k \, \phi_k(x) \int_0^t e^{-\lambda_k (t-s)} dw_s^k,
\end{aligned}
$$

It can be shown that this formal solution $u \in L^2(\Omega \times (0,T), H_0^1(D))$ is a strong solution.

Moreover it is an Ornstein-Uhlenbeck process in $H_0^1(D)$ with mean $\mathbb{E}u(x,t) = \hat{u}(x,t)$ and the covariant function

$$
\begin{aligned}
& Cov.\{u(x,t), u(y,s)\} \\
& = \sum_{k=1}^{\infty} \frac{\sigma_k^2}{2\lambda_k} \{e^{-\lambda_k |t-s|} - e^{-\lambda_k (t+s)}\} \phi_k(x) \phi_k(y).
\end{aligned}
$$

Existence of Solutions

Existence of Mild Solutions

$$\begin{aligned} du_t &= [Au_t + F_t(u_t)]dt + \Sigma_t(u_t)dW_t, \quad t \in (0,T), \\ u_0 &= h \in \mathscr{H}. \end{aligned} \tag{7}$$

Then the integral equation for a mild solution takes the form:

$$\begin{aligned} u_t &= G_t h + \int_0^t G_{t-s}F_s(u_s)ds + \int_0^t G_{t-s}\Sigma_s(u_s)dW_s \\ &\doteq \Gamma_t(u). \end{aligned} \tag{8}$$

For $p \geq 2$, let $\mathscr{B}$ be a Banach space of $\mathscr{F}_t$-adapted, continuous processes in $\mathscr{H}$ with norm: $\|u\|_{p,T} = \{\mathbb{E} \sup_{0\leq t\leq T} \|u_t\|^p\}^{1/p}$.

Under suitable conditions, show that the map $\Gamma : \mathscr{B} \to \mathscr{B}$ is a contraction. The existence of a unique solution $u \in \mathscr{B}$ follows from the Banach Fixed Point Theorem.

Existence of Strong Solutions

For example, consider the linear SPDE:

$$\frac{\partial u}{\partial t} = \nu \triangle u + b(x)u + \sigma u \frac{\partial}{\partial t} W(x,t),$$

$$u|_{\partial D} = 0, \quad u(x,0) = h(x),$$

where $b(x)$ is bounded and continuous on D; ν, σ are positive constants and $W(.,t)$ is a Wiener random field in $\mathscr{H}$. Let $\mathscr{V} = H_0^1(D), \mathscr{H} = L^2(D)$ and $\mathscr{V}^* = H^{-1}(D)$. Rewrite the above as an Itô equation:

$$du_t = Au_t + \sigma u_t dW_t, \quad u_0 = h \in \mathscr{H},$$

where $A = (\nu \triangle + b) : \mathscr{V} \to \mathscr{V}^*$ and W_t is a R-Wiener process in $\mathscr{H}$: $\mathbb{E} W_t = 0, \quad \mathbb{E}\{W_t(x) W_s(y)\}(t \wedge s) r(x,y)$ and
$(Rh)(x) = \int_D r(x,y) h(y) dy, \quad$ for $h \in \mathbb{H}$.

Proof of Existence Theorem

(1). **Galerkin Approximation**: Let $\{\phi_n\}$ a complete ONS for $\mathscr{H}$ with $\phi_n \in \mathscr{D}(A)$.

Let $P_n : \mathscr{H} \to Span\{\phi_1, \phi_2, \cdots, \phi_n\} = \mathscr{H}_n$ defined by $P_n h = \sum_{k=1}^{n} (h, \phi_n)\phi_n$, for $h \in \mathscr{H}$.

Let $A_n = P_n A$ and $W_t^n = P_n W_t$. Consider the n-dimensional Itô equation:

$$du_t^n = A_n u_t^n dt + \sigma u_t^n dW_t^n, \quad u_0^n = h_n \in \mathscr{H},$$

which has a unique solution $u^n \in L^2(\Omega \times (0,T), \mathscr{V}) \cap L^2(\Omega, C([0,T], L^2(\mathscr{H})))$.

(2). **Bounded Solutions**: From Itô's formula, B-G-D inequality and others, it can be shown that

$$\mathbb{E} \sup_{0 \le t \le T} \|u_t^n\|^2 + \mathbb{E} \int_0^T \|u_t^n\|_1^2 dt \le M_T < \infty.$$

Hence there exists a subsequence $\{u_t^k\}$ such that $u^k \rightsquigarrow u \in L^2(\Omega \times (0,T), \mathscr{V}) \cap L^2(\Omega, L_\infty((0,T), \mathscr{H}))$.

(3). **Strong Solution** For $\varphi \in \mathscr{V}$, consider the equation

$$(u_t^n, \varphi) = (h_n, \varphi) + \int_0^t (A_n u_s^n, \varphi) ds + \int_0^t (\varphi, \sigma u_s^n dW_s^n),$$

As $n \to \infty$,it will converge termwise to the variational equation a.s. :

$$(u_t, \varphi) = (h, \varphi) + \int_0^t (A u_s, \varphi) ds + \int_0^t (\varphi, \sigma u_s dW_s).$$

Properties of Solutions

Asymptotic Analysis of Solutions

$$du_t = Au_t dt + F(u_t)dt + \Sigma(u_t)dW_t, \quad u_0 = h \in \mathscr{H}$$

- **Boundedness of Solutions**: The solution u_t^h is bounded (non explosive) if, for any $T > 0$,

$$\lim_{R\to\infty} \mathbb{P}\{\sup_{0\le t\le T} \|u_t^h\| > R\} = 0.$$

- **Stability of Null Solution**: u_t^h is a.s. asymptotically stable if $\exists\, \delta > 0$ such that, for $\|h\| < \delta$, $\mathbb{P}\{\limsup_{t\to\infty} \|u_t^h\| = 0\} = 1$.

- **Existence of Invariant Measure** μ :

Let $\mu_t(B) = \mathbb{P}(u_t^h \in B | u_0^h = h)$ for $B \subset \mathscr{H}$. Show that

$$\lim_{t\to\infty} \int_{\mathscr{H}} \Phi(v)\mu_t(dv) = \int_{\mathscr{H}} \Phi(v)\mu(dv),$$

for any $\Phi \in C_b(\mathscr{H})$.

For $\varepsilon > 0$, consider

$$\begin{aligned} du_t^\varepsilon &= A u_t^\varepsilon dt + F(u_t^\varepsilon)dt + \varepsilon\Sigma(u_t^\varepsilon)dW_t, \quad t > 0, \\ u_0^\varepsilon &= h, \end{aligned}$$

• Small perturbation theory,

• Method of averaging (Multiple scales),

• Large deviations theory: Let $\mathbb{P}^\varepsilon$ be the solution measure on $X = C([0, T] \times \mathscr{H})$. Show there is a rate function J on X such that, as $\varepsilon \to 0$,

$$\varepsilon^2 log\{\mathbb{P}^\varepsilon(B)\} \sim -inf_{v \in B} J(v), \quad v \in X.$$

....

Beyond Existence THeorems:
Quantification of Solutions

Numerical Solutions

Zakai's equation in nonlinear filtering: the conditional probability density function $u(x, t, \omega)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}[a_{jk}(x)\frac{\partial u}{\partial x_k}] + \sum_{k=1}^{d} \frac{\partial}{\partial x_k}[g_k(x)u] \\ &\quad + \dot{V}(x,t)u, \quad x \in \mathbf{R}^d, t > 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{9}$$

where $V(x,t) = \sum_{k=1}^{d} h_j(x)w_j(t)$ and $w_j(t)'s$ are i.i.d Brownian motions.

Statistics of Solutions

$$\frac{\partial u}{\partial t} = \nu \triangle u + f(x,u) + \frac{\partial}{\partial t} W(t,x),$$
$$u|_{\partial D} = 0, \quad u(o,x) = h(x),$$

where $f(x,u)$ is bounded and smooth, such as $b(x)\,u$.

How to determine the statistic $\mathbb{E}\{\|u_t^h\|^2\} = \int_{\mathscr{H}} \|v\|^2 \mu_t^h(dv)$?

• **Method of Differential Equation** :

Rewrite in the equation in Itô's form

$$du_t = A u_t dt + F(u_t) dt + dW_t, \quad u_0 = h,$$

Let $\Psi : \mathscr{H} \to \mathrm{R}$ be smooth. Determine $\mathbb{E}\{\Psi(u_t^h)\} =?$

Define $\Phi(t,v)=\mathbb{E}\{\Psi(u_t^v)|u_0=v\}, v\in H$.
Then Φ satisfies the Kolmogorov equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Phi(t,v) = \frac{1}{2}Tr\,[R\,D_v^2\Phi(t,v)] \\
&+\langle Av, D_v\Phi(t,v)\rangle + (F(v), D_v\Phi(t,v)), \\
&\Phi(0,v) = \Psi(v), \qquad a.e.\ v\in\mathscr{H}.
\end{aligned}
$$

If the solution $\Phi(t,v)$ can be found, then the statistic
$\mathbb{E}\{\Psi(u_t)\}=\Phi(t,h)$.

Determination of Solution Measures

In the care of a simple stochastic heat equation, by direct computations, it is possible to determine that the solution is an O-U process in $H_0^1(D)$ with known mean and covariance. This is a vary special case. Consider

$$\frac{\partial u}{\partial t} = \nu \triangle u + b(x)u + \frac{\partial}{\partial t} W(x,t),$$
$$u|_{\partial D} = 0, \quad u(x,0) = h(x),$$

where $b(x)$ is bounded and continuous. Define the characteristic functional $\Phi(t,\lambda) = \mathbb{E}\{\exp\{i(u_t,\lambda)\}, \lambda \in \mathscr{H}$. Then Φ satisfies the Hopf's equation:

$$\Phi(t,\lambda) = \Phi(0,\lambda) + \int_0^t \{\frac{1}{2} Tr[R\, D_\lambda^2 \Phi(s,\lambda)]$$
$$+\nu\langle \Delta\lambda, D_\lambda \Phi(s,\lambda)\rangle + (b\lambda, D_\lambda \Phi(s,\lambda))\} ds.$$

Thank you