Unstable entropies and pressure of partially hyperbolic diffeomorphisms

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Entropy

- Let M be a closed Riemannian manifold, f a $C^r(r \geq 1)$ diffeomorphism on M and μ an f-invariant measure.
- Topological entropy:

$$h_{\text{top}}(f) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon),$$

where $s(n, \epsilon)$ is the cardinality of the maximal (n, ϵ) separated sets.

- Measure-theoretic entropy:

$$\begin{aligned} h_\mu(f) : &= \sup_{\alpha:\text{ finite partition}} h_\mu(f, \alpha) \\ &= \sup_{\eta:\ \text{countable measurable partition with finite entropy}} h_\mu(f, \eta) \end{aligned}$$

Variational principle

- For finite partition α,

$$h_\mu(f,\alpha) := \lim_{n\to\infty} -\frac{1}{n} \sum_{A\in\bigvee_{i=0}^{n-1} f^{-i}\alpha} \mu(A)\log\mu(A).$$

- For a countable measurable partition η,

$$h_\mu(f,\eta) := H_\mu(\eta \mid \bigvee_{i=1}^{\infty} f^{-i}\eta),$$

where, for any measurable α and η,

$$H_\mu(\alpha|\eta) = \int_M -\log \mu_x^\eta(\alpha(x))d\mu(x).$$

- Variational principle:

$$h_{\text{top}}(f) = \sup_{\mu\in\mathcal{M}(f)} h_\mu(f).$$

Lyapunov exponents and MET (Liao, Oseledec, 1960s)

Let $f \in \mathrm{Diff}^1(M)$ and μ be an f-invariant measure. There exists an invariant set Γ with $\mu(\Gamma) = 1$ and numbers Lyapunov exponents)

$$\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$$

such that

$$T_x M = \bigoplus_{i=1}^{r(x)} E_i(x) \text{ with } Df(x)E_i(x) = E_i(f(x)), \quad x \in \Gamma;$$

and

$$\lim_{n \longrightarrow \pm\infty} \frac{1}{n} \log \|Df^n(x)v\| = \lambda_i(x), \quad v \in E_i(x) \setminus \{0\}.$$

In particular, if μ is ergodic, then $\lambda_i(x)$ and $d_i(x) := \dim E_i(x)$ are constants.

The relation between entropy and Lyapunov exponents

- Ruelle, 1978: When f is C^1,
$$h_\mu(f) \leq \int \sum_{\lambda_i(x)>0} \lambda_i(x) d_i(x) d\mu.$$
- Pesin, 1977: f is C^2 and $\mu \ll \text{Leb} \Longrightarrow$ entropy formula holds,

  i.e.,
$$h_\mu(f) = \int \sum_{\lambda_i(x)>0} \lambda_i(x) d_i(x) d\mu.$$
- Mañé, 1981: f is $C^{1+\alpha}$ and $\mu \ll \text{Leb} \Longrightarrow$ entropy formula holds.
- Ledrappier and Strelcyn, 1982:

  f is $C^{1+\alpha}$ and μ is SRB $\Longrightarrow$ entropy formula holds.
- Ledrappier and Young, Ann. Math., 1985: f is C^2 (or $C^{1+\alpha}$),

  Entropy formula holds $\Longleftrightarrow$ μ is an SRB measure.

Ledrappier-Young Formula

Assume μ is ergodic.

- For μ-a.e. $x \in \Gamma$, let $\lambda_1 > \lambda_2 > \cdots \lambda_{\tilde{u}} > 0 \geq \lambda_{\tilde{u}+1} > \cdots \lambda_r$ be the distinct Lyapunov exponents. Let $W^i(x)$ be the ith unstable manifold at $x \in \Gamma$, $1 \leq i \leq \tilde{u}$.
- Let h^i denote the entropy along the W^i-foliation:

$$h_i = \lim_{\epsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_x^{\xi_i}(B_n^i(x, \epsilon)).$$

- Let δ_i denote the dimension of conditional measure on W^i:

$$\delta_i = \lim_{\epsilon \to 0} \frac{\log \mu_x^{\xi_i}(B^i(x, \epsilon))}{\log \epsilon}.$$

Theorem (Ledrappier-Young 1985, Ann. Math)

1. $h_1 = \lambda_1 \delta_1$;
2. $h_i - h_{i-1} = \lambda_i(\delta_i - \delta_{i-1})$ *for any* $1 \le i \le \tilde{u}$ *(setting* $\delta_0 = 0$*)*;
3. $h_{\tilde{u}} = h_\mu(f)$.

In particular, $h_\mu(f) = \sum_{j=1}^{\tilde{u}} \lambda_i \gamma_i$*, where* $\gamma_i = \delta_i - \delta_{i-1}$.

A question

Question

Can one define entropies, including topological entropy and measure-theoretic entropy, only along the unstable manifolds, and obtain a variational principle relating them?

Partially hyperbolic diffeomorphisms

- f is said to be partially hyperbolic if there exist an f-invariant splitting
$$TM = E^s \oplus E^c \oplus E^u$$
and numbers $0 \leq \lambda^s < 1 < \lambda^u$ such that
(1) $Df|_{E^s}$ is contracting, i.e., $\|Dfv^s\| \leq \lambda^s \|v^s\|$;
(2) $Df|_{E^u}$ is expanding, i.e., $\|Dfv^u\| \geq \lambda^u \|v^u\|$;
(3) $Df|_{E^c}$ is intermediate, i.e., $\|T_x fv^s\| < \|T_x fv^c\| < \|T_x fv^u\|$.

- Classical examples:
(1) Time-1 map of Anosov flow and frame flows.
(2) Direct product: Anosov$\times R_\theta$: $M \times \mathbb{S}^1 \longrightarrow M \times \mathbb{S}^1$.
and their perturbations.

Increasing partitions subordinate to unstable foliations

- Basic assumptions: Let f be a partially hyperbolic diffeomorphism and μ an ergodic invariant measure.
- Sinai, Pesin, Ledrappier and Young, 1980s: There exists an increasing partition ξ subordinate to $\mathcal{W}^u$, i.e.,

  (1) $f^{-1}\xi \geq \xi$;

  (2) For μ-a.e. x, $\exists r_x > 0$ such that $\xi(x)$ contains an open ball of radius r_x in $W^u_{\text{loc}}(x)$.

  Denote

  $$\mathcal{Q}^u = \{\xi \mid \xi \text{ is an increasing partition subordinate to } \mathcal{W}^u\}.$$

- An important fact: For a general C^2 diffeomorphism f and any increasing partition ξ which is subordinate to $\mathcal{W}^u$, we have

  $$h_\mu(f,\xi) = H_\mu(f^{-1}\xi|\xi).$$

Construction of increasing partitions

We recall a construction of $\xi \in \mathcal{Q}^u$ due to Ledrappier-Strelcyn.

1. Take $x \in M$ such that $\mu(S(x,r)) > 0$ for any $r > 0$, where $S(x,r) = \bigcup_{y \in W(x,r)} W^u(y,r)$. Then define a partition $\hat{\xi}_x$ such that $\hat{\xi}_x(y) = W^u(\bar{y}, r)$ if $y \in S(x,r)$, where $\bar{y} \in W(x,r)$ and $y \in W^u(\bar{y}, r)$, and $\hat{\xi}_x(y) = M \setminus S(x,r)$ otherwise. Next take $\xi = \xi_x := \vee_{j \geq 0} f^j \hat{\xi}_x$. It has been proven that for almost every small $r > 0$, ξ is subordinate to unstable manifolds W^u. Thus $\xi \in \mathcal{Q}^u$.
2. The elements of $\xi \in \mathcal{Q}^u$ can have arbitrarily small diameter, and then the object like $H(\xi|\eta)$ might not be finite. We overcome the difficulty by approximating ξ by a sequence $\{\xi_k\} := \vee_{j \leq k} f^j \hat{\xi}_x$.

More general partitions

- Let

$$\mathcal{P} = \{\alpha \mid \alpha \text{ is finite partition with } \operatorname{diam}(\alpha) < \varepsilon_0\}.$$

- For each $\alpha \in \mathcal{P}$, let

$$\eta = \{\eta(x) := \alpha(x) \cap W^u_{\text{loc}}(x) \mid x \in M\}$$

and

$$\mathcal{P}^u = \{\eta \mid \eta \text{ is obtained as above}\}.$$

- It is clear that if $\eta \in \mathcal{P}^u$ is obtained by α with $\mu(\partial\alpha) = 0$, then it is a measurable partition which is subordinate to unstable manifolds. However, it is usually not increasing.

Definition of $h_\mu^u(f)$

- For $\alpha \in \mathcal{P}, \eta \in \mathcal{P}^u$, let

$$h_\mu(f, \alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta)$$

and

$$h_\mu(f|\eta) = \sup_{\alpha\in\mathcal{P}} h_\mu(f, \alpha|\eta).$$

- The unstable metric entropy of f is defined as

$$h_\mu^u(f) = \sup_{\eta\in\mathcal{P}^u} h_\mu(f|\eta).$$

Properties of $h^u_\mu(f)$

Theorem A (Hu, Hua and Wu, Adv. Math. 2017)

For any $\alpha \in \mathcal{P}, \eta \in \mathcal{P}^u$ *and* $\xi \in \mathcal{Q}^u$,
$h_\mu(f, \alpha|\eta) = h_\mu(f, \xi) := H_\mu(\xi|f\xi)$. *Hence*
$h^u_\mu(f) = h_\mu(f|\eta) = h_\mu(f, \xi)$.

Corollary

$h^u_\mu(f) \le h_\mu(f)$, *and "$=$" holds if f is $C^{1+\alpha}$, and there is no positive Lyapunov exponent in E^c at μ-a.e. $x \in M$.*

Corollary

$h^u_\mu(f) = h_\mu(f, \alpha|\eta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta)$ *for any* $\alpha \in \mathcal{P}$ *and* $\eta \in \mathcal{P}^u$.

On the proof of Theorem A

- Note that
$$h_\mu(f,\alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|f\eta)$$
and
$$h_\mu(f,\xi) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\xi_0^{n-1}|f\xi).$$
- The "size" of η is uniform, but the "size" of ξ is nonuniform.
- Therefore, to compare these two quantities with each other needs intricate techniques.

Properties of unstable entropy

Theorem (Shannon-McMillan-Breiman Theorem)

If μ is ergodic, then
$\lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) = h_\mu(f, \alpha|\eta) \qquad \mu\text{-a.e.} x \in M.$

Theorem

$\mu \mapsto h^u_\mu(f)$ *from* $\mathcal{M}_f(M)$ *to* $\mathbb{R}^+ \cup \{0\}$ *is affine and upper-semicontinuous.*

Jiagang Yang recently obtained a more general result, i.e., the upper semi-continuity of the unstable metric entropy with respect to both the invariant measures μ and the dynamical systems f, by constructing an increasing partition ξ.

Definition of $h^u_{\text{top}}(f)$ (Hu, Hua and Wu)

The unstable topological entropy of f on M is defined by

$$h^u_{\text{top}}(f) = \lim_{\delta \to 0} \sup_{x \in M} h^u_{\text{top}}(f, \overline{W^u(x,\delta)}),$$

where

$$h^u_{\text{top}}(f, \overline{W^u(x,\delta)}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s^u(\epsilon, n, x, \delta)$$

in which $s^u(\epsilon, n, x, \delta)$ is the maximal cardinality of the (n, ϵ) d^u-separated set of $\overline{W^u(x,\delta)}$.

Unstable topological entropy

A related notion is unstable volume growth introduced by Hua-Saghin-Xia 2008: $\chi_u(f) = \sup_{x\in M} \chi_u(x,\delta)$ where $\chi_u(x,\delta) = \limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{Vol}(f^n(W^u(x,\delta)))$.

Theorem (Hu-Hua-Wu 2017)

$h^u_{top}(f) = \chi_u(f)$.

Theorem (Hua-Saghin-Xia 2008, ETDS)

$h_\mu(f) \leq \chi^u(f) + \sum_{\lambda_i^c>0} \lambda_i^c m_i$.

The theorem is an immediate corollary of Ledrappier-Young Formula and our Variational Principle when f is $C^{1+\alpha}$.

Variational principle for unstable entropies

- Let
$$\mathcal{M}_f(M) = \{\mu \mid \mu \text{ is } f\text{-invariant}\}$$
and
$$\mathcal{M}_f^e(M) = \{\nu \mid \nu \text{ is } f\text{-ergodic}\}.$$

Theorem B (Hu, Hua and Wu, 2017)

Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism. Then
$$h^u_{top}(f) = \sup\{h^u_\mu(f) : \mu \in \mathcal{M}_f(M)\}.$$
Moreover,
$$h^u_{top}(f) = \sup\{h^u_\nu(f) : \nu \in \mathcal{M}_f^e(M)\}.$$

On proof of Theorem B

- To prove VP, especially the inequality

$$h^u_{\text{top}}(f) \leq \sup\{h^u_\mu(f) : \mu \in \mathcal{M}_f(M)\},$$

we adapt the classical method of Misiurewicz to our case.

- Take a local leaf $\overline{W^u(x,\delta)}$ so that the entropy on it approximates the unstable entropy. Then take an (n,ϵ) u-separated set E_n of $\overline{W^u(x,\delta)}$ and a measure ν_n equidistributed on it. Then take an accumulation point of $\mu_n := \sum_{i=1}^{n-1} f^i_* \nu_n$.
- A key point: To ensure $\log \# E_n = H_{\nu_n}(\alpha_0^{n-1}|\eta)$, we require $\overline{W^u(x,\delta)}$ to be contained in a single element of η, i.e., $\eta(x)$. This is guaranteed if $\eta \in \mathcal{P}^u$.

Unstable topological pressure

Denote by $\mathcal{S}(n,\varepsilon)$ the set of (n,ϵ) u-separated set of $\overline{W^u(x,\delta)}$.
For $\varphi \in C(M,\mathbb{R})$, let

$$P^u(f,\varphi,\epsilon,n,\overline{W^u(x,\delta)}) = \sup\left\{\sum_{y\in E}\exp\left((S_n\varphi)(y)\right) : \ E \in \mathcal{S}(n,\varepsilon)\right\}$$

and

$$P^u(f,\varphi,\overline{W^u(x,\delta)}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P^u(f,\varphi,\epsilon,n,\overline{W^u(x,\delta)}).$$

Definition

The unstable topological pressure of f w.r.t the potential φ is defined by

$$P^u(f,\varphi) := \lim_{\delta\to 0}\sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta)}).$$

Variational principle

Theorem C (Hu, Wu and Zhu, 2018)

Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism. Then for any $\varphi \in C(M, \mathbb{R})$,

$$P^u(f, \varphi) = \sup \left\{ h^u_\mu(f) + \int_M \varphi d\mu : \mu \in \mathcal{M}_f(M) \right\}.$$

Moreover,

$$P^u(f, \varphi) = \sup \left\{ h^u_\mu(f) + \int_M \varphi d\mu : \mu \in \mathcal{M}^e_f(M) \right\}.$$

u-equilibrium and Gibbs u-states

- Let $\varphi \in C(M, \mathbb{R})$. $\mu \in \mathcal{M}_f(M)$ is called a u-equilibrium state for φ if
$$P^u(f, \varphi) = h^u_\mu(f) + \int \varphi d\mu.$$
- A Gibbs u-state is an invariant measure that has absolutely continuous conditional measures on unstable manifolds.

Theorem D (Hu, Wu and Zhu, 2018)

Let f be $C^{1+\alpha}$ and $\mu \in \mathcal{M}_f(M)$. Then μ is a Gibbs u-state of f if and only if μ is a u-equilibrium state of $\varphi^u = -\log|\det Df|_{E^u(x)}|$

Theorem D is essentially a consequence of

Lemma

If f is $C^{1+\alpha}$ and $\mu \in \mathcal{M}_f(M)$, then

$$h^u_\mu(f) \leq \int_M -\varphi^u d\mu.$$

The equality holds if and only if μ is a Gibbs u-state of f.

Corollary

If f is $C^{1+\alpha}$, then $P^u(f, \varphi^u) = 0$.

Corollary

A Gibbs u-state always exists for any PHD.

Thank You!