Strong Continuity of Eigenvalues of Sturm-Liouville Problems in Potentials and Optimal Estimates of Eigenvalues

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Abstract

In this talk, we will present some deep understanding for eigenvalues of Sturm-Liouville problems. One is the strong continuity of eigenvalues in potentials, i.e., the continuity of eigenvalues in potentials when the weak topologies are considered for potentials. Another is to use the variational method and the limiting approach to obtain optimal lower and upper bounds for eigenvalues by using the norms of potentials. Some interesting systems of degree-two-of-freedom which are resulted from the optimal eigenvalue gaps will be introduced as an open problem.

Contents

I. Eigenvalues of Sturm-Liouville Problems

We are mainly concerned with the most classical eigenvalue problem. Given a potential $q \in L^p := L^p[0,1]$, $1 \le p \le \infty$, consider

$$y'' + (\lambda + q(x))y = 0, \quad x \in [0,1]. \tag{1}$$

With the Dirichlet boundary condition

$$y(0) = y(1) = 0, \tag{D}$$

eigenvalues of (1) are denoted by

$$\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_n^D(q) < \cdots$$

Similarly, with the Neumann boundary condition

$$y'(0) = y'(1) = 0, \qquad (N)$$

eigenvalues of (1) are

$$\lambda_0^N(q) < \lambda_1^N(q) < \cdots < \lambda_n^N(q) < \cdots$$

• General Sturm-Liouville problems with integrable potentials/weights and with general separated boundary conditions or periodic/anti-periodic boundary conditions can also be considered in a similar way.

As for the dependence of eigenvalues on potentials, a classical known result is as follows.

Theorem 1. *Let* $1 \leq p \leq \infty$ *and* n *be fixed. As nonlinear functionals,*

$$q \in (L^p, \|\cdot\|_p) \to \lambda_n^{D/N}(q) \in \mathbb{R}$$

are continuously Frechét differentiable. Here $\|\cdot\|_p = \|\cdot\|_{L^p[0,1]}$ *is the usual* L^p *norm. Moreover, the Frechét derivatives are*

$$\partial_q \lambda_n^{D/N}(q) = -|E_n^{D/N}(\cdot; q)|^2 \in (L^p, \|\cdot\|_p)^*. \tag{2}$$

(As the kernels of bounded linear functionals of $(L^p, \|\cdot\|_p)$*.)*

Here $E_n^{D/N}(x;q)$ are the corresponding eigenfunctions satisfying the normalization condition

$$\|E_n^{D/N}(\cdot;q)\|_2 = 1.$$

Remarks • When the periodic/anti-periodic boundary value problems are considered, eigenvalues are only continuous nonlinear functionals of potentials. Usually speaking, these eigenvalue functionals are not differentiable at parabolic potentials.

• Formula (2) is very useful in the Inverse Scattering Method.

Note that L^p are ∞-dim Banach spaces. An alternative choice for the topologies is the so-called weak topologies.

Definition 2. *Let $p \in [1, \infty]$. The weak topology w_p in the Lebesgue space L^p is that $q_k \to q$ iff*

$$\int_0^1 q_k u \to \int_0^1 qu \quad \forall u \in L^{p^*}. \qquad (p^* := p/p-1)$$

Example 3. *(High frequencies) The Riemann-Lebesgue lemma shows that*

$$\textit{as } k \to \infty, \qquad \varphi_k(x) := \sin k\pi x \to 0 \quad \textit{in } (L^p, w_p).$$

Example 4. *(Strong bumps) For*

$$\psi_k(x) := \begin{cases} k/\log(k+1) & \textit{for } x \in [0, 1/k) \\ 0 & \textit{for } x \in [1/k, 1], \end{cases}$$

one has $\psi_k \to 0$ *in* $(L^1, \|\cdot\|_1)$ *as* $k \to \infty$.

Example 5. *(High frequencies + Strong bumps) Let* c_1 *and* c_2 *be non-zero constants. Then*

$$c_1\varphi_k + c_2\psi_k \to 0$$

in and only in the space (L^1, w_1).

Since 2008, we have revealed that eigenvalues have a very strong continuous dependence on potentials/weights. See

Zhang, *Sci. China Ser. A*, 2008

Meng & Zhang, *Acta Math. Sinica Engl. Ser.*, 2010

Yan & Zhang, *Trans. Amer. Math. Soc.*, 2011.

Theorem 6. *Given p and n, as nonlinear functionals,*

$$q \in (L^p, w_p) \to \lambda_n^{D/N}(q) \in \mathbb{R}$$

are continuous.

Eigenvalues are strong continuous in potentials!

Very few results on strong continuity. A few remarks are as follows.

• In Pöschel & Trubowitz (*Inverse Spectral Theory*, Academic Press, NY, 1987), it has been proved that Dirichlet eigenvalues $\lambda_n^D(q)$ are strong continuous in $q \in (L^p, w_p)$, where $2 \leq p \leq \infty$.

• Theorem 6 shows that perturbations of High frequencies and Strong bumps are not important to compute the specified eigenvalues.

- Theoretical explanation of Theorem 6:

Micro-quantities $q \overset{\mathcal{F}}{\mapsto}$ (nonlinear) Fourier transformations $\hat{q}$

$$\overset{\mathcal{M}}{\mapsto} \text{Macro-quantities } \lambda_n(q)$$

$$\mathcal{F} \text{ is strong continuous and } \mathcal{M} \text{ is continuous}$$
$$\Longrightarrow \mathcal{M} \circ \mathcal{F} \text{ is strong continuous}$$

Proofs: 1. Solutions of initial value problems of linear systems of ODE are strong continuous in coefficients.

2. Arguments of planar linear ODE are strong continuous.

3. Eigenvalues are obtained by solving equations for arguments. It is important to use the lefting of families of circle homeomorphisms.

The proofs are non-trivial. One way is based on the characterization of relatively compact subsets in weak topologies.

Lemma 7. *A subset* $W \subset L^1$ *is relatively compact in* (L^1, w_1) *iff*

- *(uniform boundedness)* $\sup_{u \in W} \|u\|_1 < \infty$,
- *(equi-absolute continuity) for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\left| \int_B u \right| < \epsilon \quad \forall u \in W, \ \forall B \subset [0, 1] \text{ with } \mathrm{meas}(B) < \delta.$$

For 1D linear ODE

$$y' = q(x)y, \quad x \in [0,1],$$

the solutions of initial value problems are

$$y(x;q) = y_0 \exp\left(\int_0^x q\right).$$

When y_0 and x are fixed, it follows from the definition of weak topologies that

$$q_k \to q \text{ in } (L^1, w_1) \Longrightarrow y(x;q_k) \to y(x;q).$$

By using Lemma 7, one can obtain the following strong continuity:

$$q_k \to q \text{ in } (L^1, w_1) \Longrightarrow y(\cdot; q_k) \to y(\cdot; q) \text{ in } (C^0[0,1], \|\cdot\|_{C^0}).$$

• Such a strong continuity can be obtained for solutions of initial value problems of linear systems of ODE.

• Dynamics quantities like rotation numbers and Lyapunov exponents (of Hill's equations) are also strong continuous in potentials.

• Further strong continuity results will be mentioned later.

II. Optimal Estimates of Eigenvalues of Sturm-Liouville Problems

Let $q \in L^p$, $1 \le p \le \infty$. The problem is how to estimate $\lambda_n(q) = \lambda_n^{D/N}(q)$ using the L^p norm $\|q\|_p$ of the potential q.

- Trivial result: $p = \infty$. If $\|q\|_\infty = r$, one has

$$(n\pi)^2 - r \le \lambda_n(q) \le (n\pi)^2 + r,$$

which are optimal for potentials $q \in L^\infty$.

• Case $p \in [1, \infty)$: Suppose that we have known only $\|q\|_p = r$. In order to obtain the optimal lower and upper bounds of $\lambda_n(q)$, we are lead to the following basic extremal problems

$$\mathbf{L}_{n,p}(r) := \inf_{q \in B_p[r]} \lambda_n(q) \equiv \inf_{q \in S_p[r]} \lambda_n(q), \tag{3}$$

$$\mathbf{M}_{n,p}(r) := \sup_{q \in B_p[r]} \lambda_n(q) \equiv \sup_{q \in S_p[r]} \lambda_n(q). \tag{4}$$

Here

$$B_p[r] = \{q \in L^p : \|q\|_p \le r\}, \quad S_p[r] = \partial B_p[r].$$

Of particular interest is the case $p = 1$.

Roughly speaking, by considering $q(x)$ as densities, problems (3) and (4) are predicting the ranges of oscillation frequencies when the L^p norms of $q(x)$ are known.

Problems (3) and (4) are well imposed. Once these have been solved, we will have the following optimal bounds for eigenvalues

$$\mathbf{L}_{n,p}(\|q\|_p) \leq \lambda_n^{N/D}(q) \leq \mathbf{M}_{n,p}(\|q\|_p) \quad \forall q \in L^p.$$

However, several difficulties for problems (3) and (4) include

• ∞-dim optimization problems, with the eigenvalue functionals $\lambda_n(q)$ being defined in an implicit way.

• For case $p = 1$, topologically, balls $B_1[r]$ have no compactness, even in (L^1, w_1), and, geometrically, spheres $S_1[r] = \partial B_1[r]$ are not smooth in $(L^1, \| \cdot \|_1)$.

Our approaches: Case $p \in (1, \infty)$ and case $p = 1$.

- Case $p \in (1, \infty)$: Solved by the variational method.

Topologically, $B_p[r]$ is compact and sequentially compact in the weak topology (L^p, w_p). Geometrically, $S_p[r] = \partial B_p[r]$ is a continuously differentiable manifold in $(L^p, \|\cdot\|_p)$.

The strong continuity of eigenvalues in potentials $\Longrightarrow$

$$\begin{aligned} \mathbf{L}_{n,p}(r) &:= \min\nolimits_{q \in B_p[r]} \lambda_n(q) = \lambda_n(q_{n,r,p}), \\ \mathbf{M}_{n,p}(r) &:= \max\nolimits_{q \in B_p[r]} \lambda_n(q) = \lambda_n(\hat{q}_{n,r,p}), \end{aligned} \tag{5}$$

are attained by some potentials $q_{n,r,p},\ \hat{q}_{n,r,p} \in S_p[r]$.

Using the Lagrange multiplier method, the minimizing/ maximizing potentials (on $S_p[r]$) of $\lambda_n(\cdot)$, or more general, the critical potentials, denoted by $q_p = q_{p,n,r}$, are determined by

$$E_p^2 = c_p|q_p|^{p-2}q_p,$$
$$E_p'' + (\mu_p + q_p)E_p = 0.$$

Here $c_p \neq 0$ is the Lagrange multiplier, E_p is the corresponding normalized eigen-function (and therefore satisfies the boundary condition), and $\mu_p = \lambda_n(q_p) \in \mathbb{R}$ is the value we want to characterize.

Tricky reduction: Choose the 'coordinates' for critical potentials q_p as

$$y_p := |c_p|^{-1/2} E_p,$$

a special eigen-function. Then $y = y_p$ satisfies the following critical equation

$$y'' + \mu_p y \pm |y|^{2p^*-2} y = 0, \qquad (6)$$

with the choice $+$ for min, $-$ for max. Moreover, the potential q_p is given by

$$q_p = +|y_p|^{p^*-2} y_p \quad \text{(min)}, \qquad q_p = -|y_p|^{p^*-2} y_p \quad \text{(max)}. \qquad (7)$$

Remark *The PDE counterpart of Eq. (6) is the Stationary nonlinear Schrödinger equation*

$$-\Delta u + \mu u \pm u^3 = 0 \qquad (p = 2),$$

or, more generally,

$$-\Delta u + (\mu + q_0(x))u \pm u^3 = 0 \qquad (p = 2),$$

with the ball centered at q_0*.*

By analyzing the phase portraits of critical equations (6), the minimizing/ maximizing potentials $q_p(x)$ and the extremal values μ_p can be constructed using singular integrals.

- Case $p = 1$: The limiting approach $p \downarrow 1$.

Hölder inequality $\Longrightarrow$ The smaller the exponent p is, the bigger the ball $B_p[r]$ is.

As $p \downarrow 1$, $B_p[r]$ can approximate $B_1[r]$ from the interior in the L^1 topology $\|\cdot\|_{L^p}$. Therefore

$$\mathbf{L}_{n,1}(r) = \lim_{p \downarrow 1} \mathbf{L}_{n,p}(r), \quad \mathbf{M}_{n,1}(r) = \lim_{p \downarrow 1} \mathbf{M}_{n,p}(r). \tag{8}$$

The extremal values are as follows. Denote $\mathbf{L}_n(r) := \mathbf{L}_{n,1}(r)$ and $\mathbf{M}_n(r) := \mathbf{M}_{n,1}(r)$.

Theorem 8. (Zhang, *JDE*, 2009) *Let* $\hat{\mathbf{Z}}_0 : (-\infty, 0] \to [0, \infty)$ *be*

$$\hat{\mathbf{Z}}_0(x) = \sqrt{-x}\tanh\sqrt{-x} \text{ for } x \in (-\infty, 0].$$

Then

$$\mathbf{L}_0^N(r) = \hat{\mathbf{Z}}_0^{-1}(r), \qquad \mathbf{M}_0^N(r) = r.$$

Theorem 9. (Wei-Meng-Zhang, *JDE*, 2009) *Let* $\mathbf{Z}_1 : (-\infty, \pi^2] \to [0, \infty)$ *be*

$$\mathbf{Z}_1(x) = \begin{cases} 2\sqrt{-x}\coth(\sqrt{-x}/2) & \text{for } x \in (-\infty, 0), \\ 4 & \text{for } x = 0, \\ 2\sqrt{x}\cot(\sqrt{x}/2) & \text{for } x \in (0, \pi^2], \end{cases}$$

and $\mathbf{Y}_1 : [0, \infty) \to [\pi^2, \infty)$ *be*

$$\mathbf{Y}_1(x) = \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4x}\right)^2.$$

Then, for $n \in \mathbb{N}$*,*

$$\mathbf{L}_n^{N/D}(r) = n^2\mathbf{Z}_1^{-1}(r/n^2), \qquad \mathbf{M}_n^{N/D}(r) = n^2\mathbf{Y}_1(r/n^2).$$

Remark The approach is different from that in the works of Krein, Karaa, Lou-Yanagida, ……

III. Eigenvalues for Measure Differential Equations

By considering the first Dirichlet eigenvalues $\lambda_1^D(q)$, for $p \in (1, \infty)$, let $q_p = q_{p,r} \in S_p[r]$ and $\hat{q}_p = \hat{q}_{p,r} \in S_p[r]$ be the minimizing and maximizing potentials

$$\mathbf{L}_{1,p}(r) = \lambda_1^D(q_p), \qquad \mathbf{M}_{1,p}(r) = \lambda_1^D(\hat{q}_p).$$

Lemma 10. *As $p \downarrow 1$, one has*

$$\hat{q}_{p,r} \to Q_r \quad \textit{in } (L^1, \|\cdot\|_1),$$

where $Q_r \in L^1$ is a bang-bang potential

$$Q_r(x) = \begin{cases} -\mathbf{M}_1(r) & x \in [0, \tau_r] \cup [1 - \tau_r, 1], \\ 0 & x \in [\tau_r, 1 - \tau_r]. \end{cases}$$

On the other hand, as measures,

$$Q_{p,r}(x) := \int_{[0,x]} q_{p,r}(s) ds \to r\delta_{1/2}(x) \quad \text{in } (\mathcal{M}_0[0,1], w_*),$$

where $\delta_{1/2}$ is the Dirac measure located at $1/2$ and w_* is the weak* topology in the space $\mathcal{M}_0[0,1]$ of measures.

Proposition 11. *By the continuity of eigenvalues* $\lambda_1(q)$ *in* $q \in (L^1, \|\cdot\|_1)$, *one has*

$$\mathbf{M}_1(r) = \lambda_1^D(Q_r),$$

and Q_r *is the maximizing potential for* $\lambda_1^D(q)$ *on* $S_1[r]$.

On the other hand, the minimum $\mathbf{L}_1(r)$ cannot be attained by any potential on $S_1[r]$. To explain the minimizers, we need to expand the eigenvalue theory of Sturm-Liouville problems to Measure Differential Equations (MDE). See, e.g., Meng & Zhang, *J. Diff. Eq.*, 2013.

Given a real measure $\mu \in \mathcal{M}_0[0,1]$, consider the second-order linear MDE

$$dy^{\bullet} + \lambda y dx + y d\mu(x) = 0, \quad x \in [0,1]. \tag{9}$$

With the Dirichlet boundary condition

$$y(0) = y(1) = 0, \tag{D}$$

or the Neumann boundary condition

$$y^{\bullet}(0) = y^{\bullet}(1) = 0, \tag{N}$$

problem (9) defines eigenvalues

$$\lambda_1^D(\mu) < \lambda_2^D(\mu) < \cdots < \lambda_n^D(\mu) < \cdots$$

$$\lambda_0^N(\mu) < \lambda_1^N(\mu) < \cdots < \lambda_n^N(\mu) < \cdots$$

Theorem 12. (Meng & Zhang) *As nonlinear functionals,*

$$\mu \in (\mathcal{M}_0[0,1], w_*) \to \lambda_n^{D/N}(q) \in \mathbb{R}$$

are continuous.

Very strong continuous dependence!

Proposition 13. *One has*

$$\mathbf{L}_1(r) = \lambda_1^D(r\delta_{1/2}),$$

which is attained by a completely singular measure.

IV. Some Solved and Unsolved Problems

- Solved Problem For example, as eigenvalues satisfy

$$\lambda_n(q+c) = \lambda_n(q) - c$$

for constants $c \in \mathbb{R}$, it is reasonable to study the following problems

$$\text{Min/Max} \quad \lambda_n(q) \quad \text{s.t.} \quad \bar{q} := \int_0^1 q = 0, \quad \|q\|_{L^1} = r.$$

For these problems, the critical equations contain some additional parameter m_p (due to the constraint $\bar{q} = 0$)

$$y'' + \mu_p y \pm |y^2 - m_p|^{p^*-2}(y^2 - m_p)y = 0.$$

The limiting analysis is much complicated.
Meng-Yan-Zhang have recently solved these problems.

• Partially Solved Problem — Periodic/anti-periodic eigenvalues: We have only obtained partial results. The main reason is that $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are not continuously differentiable at those potentials such that $\underline{\lambda}_n(q) = \overline{\lambda}_n(q)$ (coexistent potentials). These are related with dynamics.

• An Unsolved Problem — Optimal Gaps between Eigenvalues: For example, find

$$\inf\,(\sup)\left\{\lambda_2(q)-\lambda_1(q) : q \in S_p[r]\right\},$$

where $1 < p < \infty$.

The critical system of these problems is

$$x_i'' + \lambda_i x_i \pm |x_1^2 - x_2^2|^{p^*} x_i = 0, \quad i = 1, 2. \qquad (10)$$

Here λ_i and x_i are the critical eigenvalues and eigenfunctions.

- Nonlinear perturbation of the linear oscillators
- Non-Hamiltonian system
- Degree-2-of-freedom
- A first integral

$$\frac{1}{2}(x_1'^2 + \lambda_1 x_1^2) - \frac{1}{2}(x_2'^2 + \lambda_2 x_2^2) \pm \frac{1}{2p^*}|x_1^2 - x_2^2|^{p^*} = c.$$

Problem Is system (10) completely integrable?

Yes, if $\lambda_1 = \lambda_2 = 0$.

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Thank you!