Introduction to vertex operator algebra

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Lie algebra

Definition

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1. bilinearity:

$$[ax+by,z]=a[x,z]+b[y,z],[z,ax+by]=a[z,x]+b[z,y]$$

   for all scalars $a,b\in\mathbb{C}$ and all element $x,y,z\in\mathfrak{g}$.

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for all scalars $a, b \in \mathbb{C}$ and all element $x, y, z \in \mathfrak{g}$.
2. skew symmetry: $[x, x] = 0$, for all $x \in \mathfrak{g}$.
3. Jacobi identity:
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$
for all $x, y, z \in \mathfrak{g}$.

Motivation

The main use of Lie algebra is in studying Lie groups. The term "Lie algebra" (after Sophus Lie) was introduced by Hermann Weyl in the 1930s.

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The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the simplest example of Lie algebras. It plays very important roles in the research of Lie algebras.

1. $\mathfrak{sl}(2,\mathbb{C}) = \{A \in M_2(\mathbb{C}) \mid \mathrm{tr}(A) = 0\}$.
2. $\mathfrak{sl}(2,\mathbb{C})$ has basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying the bracket relations

$$[h,e] = 2e,\ [h,f] = -2f,\ [e,f] = h.$$

As a vector space, the affine Lie algebra

$$\widehat{\mathfrak{sl}(2,\mathbb{C})} = \mathfrak{sl}(2,\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t,t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t. The Lie brackets are defined by the formula

$$[a \otimes t^m, b \otimes t^n] = [a,b] \otimes t^{m+n} + \mathrm{tr}(ab)m\delta_{m+n,0}c$$

for $a,b \in \mathfrak{sl}(2,\mathbb{C})$ and $m,n \in \mathbb{Z}$ and

$$[c, \widehat{\mathfrak{sl}(2,\mathbb{C})}] = 0.$$

Realization problem

1. Question: To construct some concrete space such that $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ can be realized as some kinds of operators on such space.

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2. Motivation: Lepowsky and Wilson realized that such construction might eventually shed some light on the classical Rogers-Ramanujan combinatorial identities.

Fock space and operators on Fock space

1. Fock space: $S = \mathbb{C}[y_{\frac{1}{2}}, y_{\frac{3}{2}}, y_{\frac{5}{2}}, \dots]$, polynomials in the infinitely many formal variables y_n, $n = \frac{1}{2}, \frac{3}{2}, \dots$.

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    4. The operators

$$\{1, y_n, \frac{\partial}{\partial y_n} \mid n = \frac{1}{2}, \frac{3}{2}, \ldots\}$$

span an infinite dimensional Lie algebra acting on S.

Vertex operator example: $Y(x)$

In 1978, Lepowsky and Wilson constructed an operator

$$Y(x) = \exp\Big(\sum_{n=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots} \frac{y_n}{n} x^n\Big)\exp\Big(-2 \sum_{n=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots} \frac{\partial}{\partial y_n} x^{-n}\Big),$$

where "exp" is the formal exponential series and x is another formal variable commuting with the y_n's. The operator $Y(x)$ is a well-defined formal differential operator in infinitely many formal variables, including the extra variable x.
Viewing $Y(x)$ as a generating function with respect to the formal variable x, we write

$$Y(x) = \sum_{j \in \frac{1}{2}\mathbb{Z}} A_j x^{-j},$$

thus giving a family of linear operators A_j, $j \in \frac{1}{2}\mathbb{Z}$, acting on S.

Solution to the realization problem

Theorem (Lepowsky-Wilson)

The operators

$$1,\ y_n,\ \frac{\partial}{\partial y_n}\ (n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots)\ \text{and}\ A_j\ (j \in \frac{1}{2}\mathbb{Z})$$

span a Lie algebra of operators acting on S, that is, the commutator of any two of these operators is a finite linear combination of these operators, and this Lie algebra is a copy of the affine Lie algebra $\widehat{\mathfrak{sl}(2, \mathbb{C})}$.

Formal calculus

Let x be a formal variable. For a vector space V over $\mathbb{C}$, we set

$$V[[x, x^{-1}]] = \{\sum_{n\in\mathbb{Z}} v_n x^n \mid v_n \in V\},$$

as space of formal Laurent series in variable x with coefficients in V.

We set the following subspaces of $V[[x, x^{-1}]]$:

$$V[[x]] = \{\sum_{n\in\mathbb{N}} v_n x^n \mid v_n \in V\},$$

$$V((x)) = \{\sum_{n\in\mathbb{Z}} v_n x^n \mid v_n \in V,\ v_n = 0 \text{ for } n \text{ sufficiently small}\}.$$

One of the most important formal series, formal δ-function:

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n.$$

Definition

A vertex algebra is a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot, x): \qquad & V \to \mathrm{Hom}(V, V((x))) \\ & v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \ (\text{where } v_n \in \mathrm{End}\ V) \end{aligned}$$

and equipped with a distinguished vectors $1 \in V$, called the vacuum vector, such that for $u, v \in V$, the following axioms hold:

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1. $Y(1, x) = \mathrm{id}_V$;
2. $Y(v, x)1 \in V[[x]]$, and $Y(v, x)1|_{x=0} = v_{-1}1 = v$;
3. $x_0^{-1}\delta(\frac{x_1 - x_2}{x_0})Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta(\frac{x_2 - x_1}{-x_0})Y(v, x_2)Y(u, x_1)$ $= x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})Y(Y(u, x_0)v, x_2)$ (Jacobi identity).

Vertex operator algebra I

Definition

A vertex operator algebra is a $\mathbb{Z}$-graded vertex algebra

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

such that

$$\dim V_{(n)} < \infty \text{ for } n \in \mathbb{Z},$$

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently negative}$$

(grading restriction property),

Definition

(continued) and equipped with an element $\omega \in V_{(2)}$, called conformal element, such that

$$Y(\omega, x) = \sum_{n\in\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) x^{-n-2},$$

$$[L(m), L(n)] = L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} c_V$$

where $c_V \in \mathbb{C}$ is called central charge (or rank) of V and

$$Y(L(-1)v, x) = \frac{d}{dx} Y(v, x), \text{ for } v \in V,$$

$$L(0)v = nv, \text{ for } v \in V_{(n)}.$$

Applications

VOA plays an important role in conformal field theory and related areas of physics. It has also proven useful in purely mathematical contexts such as monstrous moonshine and the geometric Langlands correspondence.

Modular function of level 1

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4. A modular function of level 1 is a meromorphic function on H invariant under $\mathrm{SL}(2,\mathbb{Z})$ and satisfying a certain regularity condition at $i\infty$.

Modular function $j(\tau)$

1. $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 +$ higher order terms, where $q = e^{2\pi i\tau}$ and $\operatorname{Im} \tau > 0$.

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2. The function $j(\tau)$ is, up to normalization, the simplest nonconstant modular function of level 1; more precisely, the modular functions of level 1 are the rational functions of $j(\tau)$.
3. The function $j(\tau)$ is an isomorphism from the quotient space $H/\mathrm{SL}(2,\mathbb{Z})$ to the complex plane, which can be thought of as the Riemann sphere minus the point at ∞.

Moonshine module $V^\natural$

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2. Infinite dimensional $\mathbb{Z}$-graded module for the largest finite simple group, the Fischer-Griess Monster $\mathbb{M}$.
3. There is a decomposition:

$$V = \coprod_{n=-1,0,1,2,3,\ldots} V_n$$

such that

$$\sum_{n=-1,0,1,2,3,\ldots} (\dim V_n) q^n = j(\tau) - 744,$$

where $q = e^{2\pi i\tau}$, τ in the upper half plane.

Definition

A group G acting on the upper half plane H is called a genus 0 group if the quotient H/G is isomorphic to the Riemann sphere with a finite number of points removed.

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Example

The group $\mathrm{SL}(2, \mathbb{Z})$ is a genus 0 group.

Definition

A function from H to $\mathbb{C}$ is called Hauptmodul for a genus 0 group G if it gives an isomorphism from $\overline{H/G}$ to the sphere taking $i\infty$ to ∞.

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Moonshine conjecture

1. Conjecture (McKay-Thompson-Conway-Norton): For every $g \in \mathbb{M}$, the Thompson series

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2. The case $g \in C_G(\sigma)$, where σ is an involution in $\mathbb{M}$, is solved by Frenkel-Lepowsky-Meurman in 1988.
3. Borcherds solved all the case in 1992 by using monster vertex algebra, which is a central-charge-26 vertex algebra.

My research

Classify certain irreducible modules of monster vertex algebra.