SPDEs and Parabolic Equations in Gauss-Sobolev Spaces

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Introduction

Brownian Motion and Heat Equation

Probability Space: $(\Omega, \mathscr{F}, P)$

Brownian Motion (Wiener Process) in $\mathbb{R}^d$: a continuous stochastic process $B_t = B(t,\omega)$, $t \geq 0, \omega \in \Omega$, with $B_0 = 0$, where $B_t = (B_t^1, B_t^2, \cdots, B_t^d)$. For $F \subset \mathbb{R}^d$,

$$P\{B_t \in F\} = \int_F p(t,y)\,dy,$$

and $p(t,x)$ is the probability density function given by

$$p(t,x) = (2\pi t)^{-d/2} \exp\{-|x|^2/2t\},\ x \in \mathbb{R}^d,\ t > 0.$$

Let $X_t = x + B_t$ and $\varphi \in C_b(\mathbb{R}^d)$. Define the conditional expectation

$$u(t,x) = \mathbb{E}\{\varphi(X_t)|X_0 = x\} = \mathbb{E}\{\varphi(x + B_t)\}$$
$$= \int_{\mathbb{R}^d} \varphi(x+y)p(t,y)dy = \int_{\mathbb{R}^d} p(t,x-y)\varphi(y)dy.$$

which satisfies the heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u, \quad u(0,x) = \varphi(x).$$

Ornstein-Uhlenbeck Process

$A=[a_{ij}]_{d\times d}, \sigma=[\sigma_{ij}]_{d\times d}$: $d\times d$- matrices.

$$SDE\ in\ \mathbb{R}^d: \quad dX_t = AX_t\,dt+\sigma\,d\,B_t, \quad X_0=x\in\mathbb{R}^d,$$

$$X_t = x+\int_0^t AX_s\,ds+\sigma B_t.$$

$$X_t=e^{tA}x+Y_t, \quad Y_t=\sigma\int_0^t e^{(t-s)A}dB_s.$$

$$u(t,x)=\mathbb{E}\{\varphi(X_t)|X_0=x\}=\mathbb{E}\,\varphi(e^{tA}x+Y_t).$$

which satisfies the Kolmogorov Equation:

$$\frac{\partial u}{\partial t}=\frac{1}{2}Tr[RD^2u]+(Ax,Du), \quad u(0,x)=\varphi(x),$$

$$Du=(\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_d}), \quad Tr[RD^2u]=\sum_{i,j}^d r_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j},$$

$$(Ax,Du)=\sum_{i,j=1}^d a_{ij}x_j\frac{\partial u}{\partial x_i},\ R=\sigma\sigma*=[r_{ij}]_{d\times d}.$$

Stochastic Heat Equation

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \triangle u + \partial_t W(t,x), \quad t > 0, \\
&u(0,x) = g(x), \quad x \in \mathscr{D} \subset \mathbb{R}^d, \\
&u(t,x) = 0, \quad x \in \partial\mathscr{D},
\end{aligned} \tag{1}
$$

$$
W(t,x) = \sum_{i=1}^{\infty} \sigma_k e_k(x) B_t^k, \quad \sum_{k=1}^{\infty} \sigma_k^2 < \infty,
$$

where $\{e_k\}$ is the orthonormal set of eigenfunctions of $(-\triangle)$ with eigenvalues $\{\lambda_k\}$, and $\{B_t^k\}$ is iid Brownian motions.
Formal Solution:

$$
u(t,x) = \sum_{k=1}^{\infty} u_t^k e_k(x),\; u_t^k = (u(t,\cdot), e_k) = \int_{\mathscr{D}} u(t,x)\, e_k(x) dx,
$$

$$
du_t^k = -\lambda_k u_t^k dt + \sigma_k dB_t^k, \quad u_0^k = g_k = \int_{\mathscr{D}} g(x)\, e_k(x) dx.
$$

$$
u_t^k = e^{-\lambda_k t} g_k + \sigma_k \int_0^t \exp\{-\lambda_k (t-s)\}\, dB_s^k.
$$

By direct verification, it can be shown that

$$u \in L^2((0,T)\times\Omega; H_0^1)\bigcap L^2(\Omega; C([0,T],H),$$

and it satisfies

$$\int_{\mathscr{D}} u(t,x)\phi(x)dx = \int_{\mathscr{D}} g(x)\phi(x)dx + \int_0^t \int_{\mathscr{D}} \triangle u(s,x)\phi(x)dx \\ + \int_{\mathscr{D}} \phi(x)dW(t,x)dx,$$

for each $\phi \in H_0^1$, where $H = L^2(\mathscr{D})$ and $H_0^1 = H_0^1(\mathscr{D})$.

Stochastic Reaction-Diffusion Equation

Initial-boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \triangle u + f(u,x) + \partial_t W(t,x), \quad t > 0, \\ &u(0,x) = g(x), \quad x \in \mathscr{D} \subset \mathbb{R}^d, \\ &u(t,x) = 0, \quad x \in \partial\mathscr{D}, \end{aligned} \tag{2}$$

where $W(t,x)$, for $x \in \mathbb{R}^d$, $t \geq 0$, be a continuous Wiener random field defined in $(\Omega, \mathscr{F}, P)$ with mean $\mathbb{E}\, W(t,x) = 0$ and covariance function $r(x,y)$ defined by $\mathbb{E}\, W(t,x)W(s,y) = (t \wedge s) r(x,y), \quad x,y \in \mathbb{R}^d$, where $(t \wedge s) = \min(t,s)$ for $0 \leq t,s \leq T$.

Fact: Suppose that $f : \mathbb{R} \times \mathscr{D} \to \mathbb{R}$ is Lip-continuous and $r(x,y)$ is bounded and continuous for $x,y \in \mathscr{D}$. Then, for each $g \in H$, $T > 0$, the problem (2) has a unique solution $u \in L^2((0,T) \times \Omega; H_0^1) \bigcap L^2(\Omega; C([0,T]; H))$.

Parabolic Itô Equation

Itô Equation in Hilbert Space H (in distributional sense)

$$\begin{aligned} du_t &= [A\,u_t + F(u_t)]\,dt + d\,W_t, \quad 0 < t < T, \\ u_0 &= v \in H, \end{aligned} \tag{3}$$

where $A : H^1 \to H^{-1}$ with domain $H_0^1 \cap H^2$, $F : H \to H$ is Lip-continuous and W_t is a $H-$valued Wiener process with a trace-class covariance operator R on H.

Fact: If A is strongly elliptic and F is Lip.-continuous, then Eq.(3) has a unique solution $u \in L^2((0,T)\times\Omega; H_0^1)\bigcap L^2(\Omega; C([0,T]; H))$, which satisfies

$$\int_{\mathscr{D}} u(t,x)\phi(x)dx = \int_{\mathscr{D}} g(x)\phi(x)dx + \int_0^t\int_{\mathscr{D}} Au(s,x)\phi(x)dx + \int_0^t\int_{\mathscr{D}} F(u(s,x))\phi(x)dx + \int_{\mathscr{D}} \phi(x)W(t,x)dx,$$

for each $\phi \in H_0^1$.

Kolmogorov Equation for SPDE – Example

$$\frac{\partial u}{\partial t} = \triangle u + \partial_t W^n(t,x), \quad 0 < x < \pi,\, t > 0,$$
$$u(0,x) = u_0^n(x), \quad u(t,0) = u(t,\pi) = 0,$$

with $W_t^n = \sum_{k=1}^n \sigma_k B_t^k e_k$ and $u_0^n = \sum_{k=1}^n g_k e_k$, where $e_k(x) = \sqrt{2}\sin kx$ is the eigenfunction of $(-\triangle)$ with eigenvalue $\lambda_k = k^2$ for $k = 1, 2, \cdots, n$. There exists a finite-dimensional solution: $u(t,\cdot) = \sum_{k=1}^n u_t^k e_k(x)$, where u_t^k is an O-U process given by

$$u_t^k = e^{-k^2 t} g_k + \sigma_k \int_0^t e^{-k^2(t-s)} d\, B_s^k, \quad k = 1, 2, \cdots, n.$$

As before, for $F \in C_b^2(H)$ with $F(v^n) = f(v_1, v_2, \cdots, v_n)$, define $\Phi(t, v^n) = \mathbb{E}\{F(u_t) | u_0 = v^n\}$.

Kolmogorov Equation for SPDE – Example

$$\frac{\partial \Phi}{\partial t} = \frac{1}{2}\sum_{i}^{n} \sigma_i^2 \frac{\partial^2 \Phi}{\partial v_i^2} - \sum_{k=1}^{n} k^2 v_k \frac{\partial \Phi}{\partial v_k},$$
$$\Phi(0, v^n) = \varphi(v^n).$$

Q: What happens as $n \to \infty$?
Formally, as $n \to \infty$, the above yields

$$\frac{\partial \Phi}{\partial t} = \frac{1}{2} Tr[RD^2\Phi] + (\triangle v, D\Phi), \qquad (4)$$
$$\Phi(0, v) = \varphi(v),$$

where $D\Phi, D^2\Phi, \cdots$ denote the Fréchet derivatives of Φ in H.
Remarks:

(1)The above equation is defined only when $v \in \mathscr{D}(\triangle) \subset H$!

(2) Clearly Eq.(4) has no classical solutions.

(3) In what sense the function $\Phi(t, v) = \mathbb{E}\{F(u_t)|u_0 = v)\}$ is a solution of Eq.(4)?

Kolmogorov Equation for Parabolic Itô Equation

$$du_t = [A u_t + F(u_t)]\, dt + d\, W_t, \quad 0 < t < T,$$
$$u_0 = v \in H,$$

where $A : H^1 \to H^{-1}$ with domain $H_0^1 \cap H^2$, $F : H \to H$ is continuous and W_t is a $H-$valued Wiener process with a trace-class covariance operator R on H.

Let $\varphi : H \to \mathbb{R}$ be a smooth function. Then

$$\Phi_t(v) = \mathbb{E}\{\varphi(u_t)|u_0 = v\}$$

satisfies the Kolmogorov equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Phi_t(v) &= \mathscr{L}\,\Phi_t(v) + (F, D\Phi_t(v)),\ v \in \mathscr{D}(A), \quad t > 0, \\ \Phi_0(v) &= \varphi(v), \end{aligned} \tag{5}$$

where $\mathscr{L}$ will be called the O-U (Ornstein-Uhlenbeck) operator defined by $\mathscr{L}\Phi(v) = \frac{1}{2} Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle$, and $D\Phi, D^2\Phi, \cdots$ denote the derivatives of Φ in H.

Stochastic Control Problem

$$du_t = [A u_t + F(u_t, \eta_t)]\, dt + d\, W_t, \quad 0 < t < T,$$
$$u_0 = v \in H,$$

where $F(\cdot, \eta_t)$ depends on the control η_t in a bounded convex set $\mathscr{K}_T$ of admissible controls.
The problem: Find $\eta^\star \in \mathscr{K}_T$ which minimizes the cost function

$$J(t, v, \eta) = \mathbb{E}\{\int_t^T e^{-\alpha t} B(u_s, \eta_s)\, ds + \varphi(u_T) | u_t = v\},$$

where $B : H \times \mathscr{K}_T \to \mathbb{R}^+$ is the running cost with the discount rate $\alpha > 0$ and $\varphi : H \to \mathbb{R}^+$ is the terminal cost.

Hamilton-Jacobi-Bellman Equation

Define the value function: $V_t(u) = \inf_{\eta \in \mathscr{K}_T} J(t, u, \eta)$.
By the dynamic programming principle, the function $\Phi_t = V_{T-t}$ satisfies the H-J-B equation:

$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi_t(u) &= (\mathscr{L} - \alpha)\Phi_t(u) + \mathscr{F}(u, D\Phi_t(u)), \quad t > 0, \\
\Phi_0(u) &= \varphi(u),
\end{aligned}
$$

where

$$
\mathscr{F}(u, D\Phi) = \inf_{\eta \in \mathscr{K}_T} \{ (F(u,\eta), D\Phi) + B(u,\eta) \}.
$$

L^2- **Gauss- Sobolev Spaces**

Theory in L^2-Sobolev Spaces

What are needed for L^2- theory?

(R.1) Choose a suitable measure μ for integration.

(R.2) Workable differential and integral calculus, such as integration by parts formula.

(R.3) Suitable function spaces for solutions.

Linear Itô Equation

H: real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$.
$V \subset H$: Hilbert subspace with norm $\|\cdot\|$.
V': the dual space of V with the duality pairing $\langle\cdot,\cdot\rangle$.
(Assume that the inclusions $V \subset H \cong H' \subset V'$ are dense and continuous.)
$A : V \to V'$: continuous closed linear operator with domain $\mathscr{D}(A)$ dense in H,
W_t : H-valued Wiener process with trace-class covariance operator R.
Consider the linear stochastic equation in a distributional sense:

$$\begin{aligned} du_t &= Au_t\,dt + d\,W_t, \quad t \geq 0, \\ u_0 &= h \in H. \end{aligned} \tag{6}$$

Assume Conditions (A):

(A.1) Let $A : V \to V'$ be a self-adjoint, coercive operator such that $\langle -Av, v\rangle \geq \beta \|v\|^2$, for some $\beta > 0$, and $(-A)$ has eigenvalues $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots$, counting the finite multiplicity, with $\alpha_n \uparrow \infty$ as $n \to \infty$. The corresponding orthonormal set of eigenfunctions $\{e_n\}$ is complete.

(A.2) The resolvent operator $\mathscr{R}_\lambda(A)$ and covariance operator R commute.

(A.3) The covariance operator $R : H \to H$ is a self-adjoint operator with a finite trace such that $R^{1/2}H \subset V$.

Then the following hold:

(1) A generates a contraction semigroup $\{e^{tA}, t \geq 0\}$ on H.

(2) The solution u_t is a Gaussian (diffusion) process in H with the transition probability $\mu_t^v(B) = P(u_t \in B | u_0 = v)$, for $v \in H$ and $B \in \mathscr{B}(H)$.

Invariant Measure

Transition Operator: For any $\Psi \in C_b(H)$, define

$$\mathscr{P}_t\Psi(v) = \int_H \Psi(\eta)\,\mu_t^v(d\eta)$$

Invariant measure μ :

$$\int_H \mathscr{P}_t\Psi(\eta)\mu(d\eta) = \int_H \Psi(\eta)\,\mu(d\eta),\ \forall\,\Psi \in C_b(H),\, t \geq 0.$$

Lemma 1.1 Under Conditions (A), we have $\mu_t^v \rightharpoonup \mu$ (weak convergence) in the sense that

$$\lim_{t\to\infty} \mathscr{P}_t\Psi(v) = \lim_{t\to\infty} \int_H \Psi(\eta)\,\mu_t^v(d\eta) = \int_H \Psi(\eta)\,\mu(d\eta),$$

for all $v \in H, \Psi \in C_b(H)$. Moreover μ is the unique invariant measure of the stochastic equation (6) , which is a centered Gaussian measure on H supported in V with covariance operator $\Gamma = \dfrac{1}{2}(-A)^{-1}R$. $\Box$

Hermite Polynormials

Let $\mathscr{H} = L^2(H, \mu)$ with norm $|||\Phi||| = \{\int_H |\Phi(v)|^2 \mu(dv)\}^{1/2}$, and inner product $[\cdot, \cdot]$ given by

$$[\Theta, \Phi] = \int_H \Theta(v)\Phi(v)\mu(dv), \quad \text{for}\, \Theta, \Phi \in \mathscr{H}.$$

Let $\mathbf{n} = (n_1, n_2, \cdots, n_k, \cdots)$, where $n_k \in \mathbb{Z}^+$, the set of nonnegative integers, and let $\mathbf{Z} = \{\mathbf{n} : n = |\mathbf{n}| = \sum_{k=1}^{\infty} n_k < \infty\}$.

Let $h_m(r)$ be the one-dimensional Hermite polynomial of degree m. For $v \in H$, define a Hermite (polynomial) functional of degree n by

$$H_{\mathbf{n}}(v) = \prod_{k=1}^{\infty} h_{n_k}[\ell_k(v)],$$

where we set $\ell_k(v) = (v, \Gamma^{-1/2} e_k)$ and $\Gamma^{-1/2}$ denotes a pseudo-inverse.

For a smooth functional Φ on H, let $D\Phi$ and $D^2\Phi$ denote the Fréchet derivatives of the first and second orders, respectively.

L^2_μ-Gauss-Sobolev Spaces

Let $\mathscr{L}$ be the O-U operator

$$\mathscr{L}\Phi(v) = \frac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle \tag{7}$$

defined for a polynomial functional Φ with $v \in \mathscr{D}(A)$.

Theorem 1.2 The set of all Hermite functionals $\{H_{\mathbf{n}} : \mathbf{n} \in \mathbf{Z}\}$ forms a complete orthonormal system in $\mathscr{H}$. Moreover we have $\mathscr{L}H_{\mathbf{n}}(v) = -\lambda_{\mathbf{n}} H_{\mathbf{n}}(v)$, $\forall \mathbf{n} \in \mathbf{Z}$, where $\lambda_{\mathbf{n}} = \mathbf{n}\cdot\alpha = \sum_{k=1}^{\infty} n_k\alpha_k$. $\square$

Let $\Phi_{\mathbf{n}} = [\Phi, H_{\mathbf{n}}]$. For any positive integer m, define

$$|||\Phi|||_m = |||(I-\mathscr{L})^{m/2}\Phi||| = \{\sum_{\mathbf{n}}(1+\lambda_{\mathbf{n}})^m|\Phi_{\mathbf{n}}|^2\}^{1/2}, \tag{8}$$

with I being the identity operator in $\mathscr{H} = \mathscr{H}_0$. Let $\mathscr{H}_m$ denote the Gauss-Sobolev space of order m defined by

$$\mathscr{H}_m = \{\Phi \in \mathscr{H} : |||\Phi|||_m < \infty\}.$$

Integration by Parts

Remarks:

(1) In particular, for $m \geq 1$, we have $\mathscr{H}_m \subset \mathscr{H} \subset \mathscr{H}_{-m}$.

(2) The norm $|||\Phi|||_1$ in $\mathscr{H}_1$ is equivalent to the norm

$$\|\Phi\|_R^1 := \{|||\Phi|||^2 + |||D_R\Phi|||^2\}^{\frac{1}{2}},$$

where $D_R\Phi = R^{\frac{1}{2}} D\Phi$ or the derivative in the direction of $\overline{\{R^{\frac{1}{2}} H\}}$.

Lemma 1.3 (Integration by Parts) For $\phi, \psi \in \mathscr{H}_1$ and $g \in (\Gamma^{1/2} H)$, the following formula holds

$$\int_H (D_R\phi, g)\psi d\mu = -\int_H (D_R\psi, g)\phi d\mu + \int_H (v, \Gamma^{-1/2} g)\phi \psi d\mu.$$

Recall, for a smooth function Φ, the O-U operator

$$\mathscr{L}\Phi(v) = \frac{1}{2} Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle.$$

Let $\mathscr{P}_N$ be a projection operator in $\mathscr{H}$ onto its subspace $\mathscr{S}_N$ spanned by the Hermite polynomial functionals of degree N. Define $\mathscr{L}_N = \mathscr{P}_N\mathscr{A}$. Then the following theorem holds.

Theorem 1.4 (Integration by Parts) The sequence $\{\mathscr{L}_N\}$ converges strongly to a linear symmetric operator $\mathscr{L} : \mathscr{H}_2 \to \mathscr{H}$, so that, for $\Phi, \Psi \in \mathscr{H}_2$, the following identity holds:

$$\int_H [\mathscr{L}\Phi, \Psi]\, d\mu = \int_H [\Phi, \mathscr{L}\Psi]\, d\mu = -\frac{1}{2}\int_H [D_R\Phi, D_R\Psi]\, d\mu. \quad (9)$$

Moreover $\mathscr{L}$ has a self-adjoint extension, still denoted by $\mathscr{L}$ with domain $\mathscr{D}(\mathscr{L}) \supset \mathscr{H}_2$. □

Solutions of Parabolic Equations

Linear Parabolic Equations

Let $F : H \to H, \mathscr{G} : H \to \mathbb{R}$ be bounded and continuous. For $Q \in L^2((0,T);\mathscr{H})$ and $\phi \in \mathscr{H}$, consider the Cauchy problem:

$$\begin{aligned} \frac{\partial}{\partial t}\Phi_t(v) &= \mathscr{L}\Phi_t(v) + (F(v), D_R\Phi_t(v)) + \mathscr{G}(v)\Phi_t(v) \\ &\quad + Q_t(v), \quad \mu - a.e.\ v \in H, \quad t \in (0,T), \\ \Phi_0(v) &= \phi(v), \end{aligned} \tag{10}$$

Strong Solution: A continuous function $\Phi : [0,T] \times \mathscr{H} \to \mathbb{R}$ is said to be a strong solution of Eq.(10) if $\Phi \in C([0,T];\mathscr{H}) \cap L^2((0,T);\mathscr{H}_1)$ and it satisfies

$$\begin{aligned} [\Phi_t, \varphi] &= [\phi, \varphi] + \int_0^t \ll \mathscr{L}\Phi_s, \varphi \gg ds + \int_0^t [(F, D_R\Phi_s), \varphi]\, ds \\ &\quad + \int_0^t [\mathscr{G}(v)\Phi_s(v), \varphi]\, ds + \int_0^t [Q_s, \varphi]\, ds, \end{aligned} \tag{11}$$

for all $\varphi \in \mathscr{H}_1$, $a.e.\ t \in [0,T]$.

Energy Estimates

Theorem 2.1 Assume that $F : H \to H, \mathscr{G} : H \to \mathbb{R}$ are bounded and continuous. Then the following inequalities hold

(1) For any $\Phi, \Psi \in \mathscr{H}_1$, there exist constants $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$, such that

$$|\ll \mathscr{L}\Phi, \Psi \gg| \leq \alpha |||\Phi|||_1 |||\Psi|||_1,$$

$$\ll \mathscr{L}\Phi, \Phi \gg \leq -\beta |||\Phi|||_1^2 + \gamma |||\Phi|||^2.$$

(2) There exists a positive constant C, depending on $F, \mathscr{G}$ and T, such that, for a smooth function $u_t(v), t \in [0, T], v \in \mathscr{H}$,

$$\sup_{0 \leq t \leq T} |||u_t|||^2 + \int_0^T |||u_s|||_1^2 \, ds + \int_0^T |||\partial_s u_s|||_{-1}^2 \, ds$$
$$\leq C\{|||u_0|||^2 + \int_0^T |||Q_s|||^2 \, ds\}. \qquad \square$$

Existence Theorem

Theorem 2.2 Suppose that $F : H \to H, \mathscr{G} : H \to \mathbb{R}$ are bounded and continuous. Then, for each $\varphi \in \mathscr{H}$ and $Q \in L^2((0,T);\mathscr{H})$, the Cauchy problem

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi_t(v) &= \mathscr{L}\Phi_t(v) + (F(v), D_R\Phi_t(v)) + \mathscr{G}(v)\Phi_t(v) \\
&\quad + Q_t(v), \quad \mu - a.e.\ v \in H, \quad t \in (0,T), \\
\Phi_0(v) &= \varphi(v)
\end{aligned}$$

has a unique strong solution $\Phi \in C([0,T];\mathscr{H}) \cap L^2((0,T);\mathscr{H}_1)$.

Lemma 2.3 The embedding $\mathscr{H}_1(H,\mu) \hookrightarrow \mathscr{H} = L^2(H,\mu)$ is compact. (Da Prato, Malliavin, Nualart, 2002) □

Idea of Proof

1. Galerkin Approximation: Show that the finite-dimensional problem

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi_t^n(v) &= \mathscr{L}_n \Phi_t^n(v) + (F_n(v), D_R^n \Phi_t^n(v)) + \mathscr{G}_n(v)\Phi_t^n(v) \\
&\quad + Q_t^n(v),\ v \in H, \quad t \in (0,T), \\
\Phi_0^n(v) &= \varphi_n(v)
\end{aligned}$$

   has a unique strong solution
   $\Phi^n \in C([0,T];\mathscr{H}) \cap L^2((0,T);\mathscr{H})$.

2. By the energy estimates and the compact embedding Lemma, show that the sequences $\{\Phi^n\}$ and $\{\dot{\Phi}^n\}$ are bounded in $L^2((0,T);\mathscr{H}_1)$ and $L^2((0,T);\mathscr{H}_{-1})$ ' respectively. So there exists a function $\Phi \in L^2((0,T);\mathscr{H}_1)$, with $\dot{\Phi} = \partial_t \Phi \in L^2((0,T);\mathscr{H}_{-1})$, and a subsequence $\{\Phi^{n_k}\}$ such that $\Phi^{n_k} \rightharpoonup \Phi \in L^2((0,T);\mathscr{H}_1)$ and $\dot{\Phi}^{n_k} \rightharpoonup \dot{\Phi} \in L^2((0,T);\mathscr{H}_{-1})$.

3. Show that the weak limit Φ is a strong solution.
   For $\psi \in \mathscr{H}_1$, Φ^{n_k}, as a strong solution, satisfies

$$\begin{aligned}[\Phi_t^{n_k}, \psi] &= [\varphi^{n_k}, \psi] + \int_0^t \ll \mathscr{L}_{n_k} \Phi_s^{n_k}, \psi \gg ds \\ &\quad + \int_0^t [(F_{n_k}, D_R \Phi_s^{n_k}), \psi]\, ds + \int_0^t [Q_s^{n_k}, \psi] ds,\end{aligned}$$

   which will converge, as $n_k \to \infty$, to

$$\begin{aligned}[\Phi_t, \psi] &= [\varphi, \psi] + \int_0^t \ll \mathscr{L} \Phi_s, \psi \gg ds \\ &\quad + \int_0^t [(F, D_R \Phi_s), \psi]\, ds + \int_0^t [Q_s, \psi] ds,\end{aligned}$$

   or the weak limit Φ is a strong solution.
4. The uniqueness follows from the energy inequality.

Nonlinear Parabolic Equations

Fundamental Solution

For $\Phi \in \mathscr{H}$, define

$\mathscr{P}_t\Phi(v) = E\{\Phi(u_t)|u_0 = v\}$. $(du_t = Au_t dt + dW_t)$

$\mathscr{R}_t\Phi = e^{-\alpha t}\mathscr{P}_t\Phi,$ for $\alpha > 0$.

Theorem 3.1 Under Conditions (A), the transition operator $\mathscr{R}_t$ is defined on $\mathscr{H}$ for all $t \geq 0$ and $\{\mathscr{R}_t : t \geq 0\}$ forms a strongly continuous semigroup of linear operators on $\mathscr{H}$ with the infinitesimal generator $\mathscr{L}_\alpha \doteq (\mathscr{L} - \alpha I)$ in $\mathscr{H}_2$. Moreover, for $\phi \in \mathscr{H}$, $Q \in L^2((0,T);\mathscr{H})$, the function $\Phi_t(v)$ defined by

$$\Phi_t(v) = \mathscr{R}_t\phi(v) + \int_0^t (\mathscr{R}_{t-s}Q_s)(v)\, ds$$

is the strong solution of the Cauchy problem

$$\frac{\partial}{\partial t}\Phi_t = \mathscr{L}_\alpha \Phi_t(v) + Q_t,\ \Phi_0 = \phi, \quad t \in (0,T). \tag{12}$$

Basic Estimates

Lemma 3.2 Let $\Phi \in \mathscr{H}_{m-1}$ and $Q \in L^2((0,T);\mathscr{H}_{m-1})$ for any integer $m \geq 0$. The following inequalities hold:

(1) $|||\mathscr{R}_t\Phi|||_m \leq e^{-\alpha t}|||\Phi|||_m$,

(2) $|||\int_0^t [\mathscr{R}_{t-s}\Phi]\, ds|||_m^2 \leq \frac{t}{2\alpha_1}|||\Phi|||_{m-1}^2$,

(3) $|||\int_0^t [\mathscr{R}_{t-s}Q_s\, ds|||_m^2 \leq \frac{1}{2\alpha_1}\int_0^t |||Q_s|||_{m-1}^2$, for $t \in [0,T]$,
where $\alpha_1 = min\{\alpha, 1\}$.

Nonlinear Parabolic Equation

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t &= \mathscr{L}_\alpha \Psi_t + \mathscr{B}(\Psi_t) + Q_t, \quad t > 0, \\ \Psi_0 &= \Theta. \end{aligned} \tag{13}$$

Assume that $\mathscr{B} : \mathscr{H}_1 \to \mathscr{H}$ is bounded and continuous such that the following conditions hold:

(B1) There exists a positive function ρ_1 on $\mathscr{H}$ with $|||\rho_1||| < \infty$ such that

$$|||\mathscr{B}(\Phi)|||^2 \leq \rho_1 \{1 + |||\Phi|||^2 + |||D_R\Phi|||^2\}, (D_R\Phi = R^{1/2}D\Phi)$$

(B2) There exists a positive function ρ_2 on $\mathscr{H}$ with $|||\rho_2||| < \infty$ such that

$$|||\mathscr{B}(\Phi) - \mathscr{B}(\Phi')|||^2 \leq \rho_2 \{ |||\Phi - \Phi'|||^2 + |||D_R(\Phi - \Phi')|||^2 \}.$$

Existence Theorem

Theorem 3.3 Suppose that $\mathscr{L}_\alpha$ is given as before, and the conditions (C1) and (C2) hold true. Then, for $\Theta \in \mathscr{H}$ and $Q \in L^2((0,T);\mathscr{H})$, the Cauchy problem (13) has a unique strong solution $\Psi \in C([0,T];\mathscr{H}) \cap L^2((0,T);\mathscr{H}_1)$, for any $T > 0$, so that the following equation holds for every $t \in [0,T]$, $\phi \in \mathscr{H}_1$:

$$[\Psi_t, \phi] = [\Theta, \phi] + \int_0^t \langle\langle \mathscr{L}_\alpha \psi_s, \Phi \rangle\rangle\, ds + \int_0^t [\mathscr{B}(\Psi_s), \phi]\, ds + \int_0^t [Q_s, \Phi]\, ds.$$

Moreover the solution satisfies the inequality:

$$\begin{aligned} &\sup_{0 \le t \le T} |||\Psi_t|||^2 + \int_0^T |||\Psi_s|||_1^2\, ds \\ &\le C_T\{1 + |||\Phi|||^2 + \int_0^T |||Q_s|||^2\, ds\}, \end{aligned}$$

for some constant $C_T > 0$, depending on T and $\mathscr{B}$.

Idea of Proof

Introduce the Banach space $X_T := C([0,T];\mathscr{H}) \cap L^2((0,T);\mathscr{H}_1)$ with the norm defined by

$$|||\Psi|||_T^2 = \sup_{0\le t\le T} |||\Psi_t|||^2 + \int_0^T |||\Psi_s|||_1^2\, ds.$$

Define the map: $\mathscr{F}_\cdot : X_T \to X_T$ by

$$\mathscr{F}_t(\Psi) := [\mathscr{R}_t\Phi] + \int_0^t \mathscr{R}_{t-s}\mathscr{B}_s(\Psi_s)ds + \int_0^t \mathscr{R}_{t-s}Q_s ds.$$

Show that the map $\mathscr{F}_\cdot$ is a contraction map in X_T with respect to an equivalent norm

$$|||\Psi|||_{\lambda,T} = \sup_{0\le t\le T} |||\Psi_t|||^2 + \lambda \int_0^T |||\Psi_s|||_1^2 ds,$$

for some $\lambda > 1$.

Stationary solutions

Linear Parabolic Equations

$$\frac{\partial}{\partial t}\Phi_t = \mathscr{L}_\alpha \Phi_t + Q_t, \qquad \Phi_0 = \Theta. \tag{14}$$

Theorem 4.1 Let $Q_t \in \mathscr{H}$ be bounded and continuous in $t \in [0, \infty)$ such that the following condition holds in $\mathscr{H}$

$$\lim_{t\to\infty} Q_t = Q. \tag{15}$$

Then under conditions (A.1)–(A.3), for any $\Theta \in \mathscr{H}$, there exists the limit

$$\lim_{t\to\infty} \Phi_t = \lim_{t\to\infty} \{\mathscr{R}_t\Theta + \int_0^t \mathscr{R}_{t-s} Q_s\, ds\} = \Psi, \tag{16}$$

and $\Psi \in \mathscr{H}_2$ satisfies the elliptic equation:

$$\mathscr{L}_\alpha \Psi = -Q. \quad \Box \tag{17}$$

Nonlinear Parabolic Equations

$$\begin{aligned} \frac{d}{dt}\Phi_t &= \mathscr{L}_\alpha \Phi_t + \mathscr{B}(\Phi_t) + Q_t, \quad t>0, \\ \Phi_0 &= \Theta. \end{aligned} \tag{18}$$

(C.1) Let $\mathscr{B}(\cdot):\mathscr{H}_1 \to \mathscr{H}$ a continuous mapping with $\mathscr{B}(0)=0$. Suppose there exist positive constants b_1, b_2, such that $b_2 < \sqrt{b_1} < \alpha$ and, for any $\phi, \psi \in \mathscr{H}_1$,

$$|||\mathscr{B}(\phi) - \mathscr{B}(\psi)|||^2 \leq b_1 |||\phi - \psi|||^2 + b_2 |||R^{\frac{1}{2}} D(\phi - \psi)|||.$$

(C.2) The map $\mathscr{B}$ can be extended to be a continuous operator from $\mathscr{H}$ into $\mathscr{H}_{-1}$ such that $|||\mathscr{B}(\phi) - \mathscr{B}(\psi)|||_{-1} \leq \kappa |||\phi - \psi|||$, for some constant $\kappa > 0$, and for any $\phi, \psi \in \mathscr{H}$.

(C.3) Q_t is a bounded continuous $\mathscr{H}$-valued function on $[0,\infty)$ such that

$$\lim_{t\to\infty} Q_t = Q.$$

Theorem 4.2 Suppose that conditions (C.1)–(C.3) hold. Then, for any $t > 0$ and $\Theta \in \mathscr{H}$, the solution Ψ_t of the Cauchy problem (18) converges to the limit:

$$\lim_{t\to\infty} \Psi_t = \Psi, \tag{19}$$

and Ψ is the mild solution of (18) which satisfies the following equation:

$$\Psi = -\mathscr{L}_\alpha^{-1}\,[\mathscr{B}(\Psi) + Q], \tag{20}$$

where $\mathscr{L}_\alpha^{-1}\mathscr{B}(\cdot) \doteq \mathscr{L}_\alpha^{-1} \circ \mathscr{B}(\cdot)$ is a bounded operator on $\mathscr{H}$. Moreover the solution of equation (20) is unique if, in condition (B.2), $\kappa < \sqrt{\alpha\alpha_1}$ with $\alpha_1 = \min\{\alpha, 1\}$. $\square$

General Remarks:

(1) For parabolic equations in infinite dimensions, there is no canonical reference measure. The measure to be chosen must be explicit and compatible with the elliptic operator in the equation.

(2) Unlike Sobolev spaces in $\mathbb{R}^n$, so far, very little is known about the properties of Gauss-Sobolev spaces. For instance, we know the embedding $\mathscr{H}_1 \subset \mathscr{H}$ is compact. But $\mathscr{H}_2 \subset \mathscr{H}_1$ is not.

(3) There exist no general Sobolev inequalities as in finite dimension. In particular it is not known how to relate a solution in a Sobolev space to one in the space of continuous functions.

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